Unit 1

Getting started

Introduction

Welcome to MST224 *Mathematical methods*. This module is designed to teach you the mathematics needed to study the physical sciences and other subjects (such as economics) where mathematics is used to model phenomena in the real world.

Everything that you will learn in this module has applications in a wide range of topics. Because the same mathematical techniques appear in different contexts, the mathematical knowledge required by a really good scientist need not be that extensive. This module will teach you most of the essential tools. If you go on to further study, you will find that while some additional mathematical ideas may be needed, most of the time you will be discovering the power of what you have learned in this module.

Because it is easier to learn just one new thing at a time, this module will concentrate on teaching the mathematics. It will not attempt to teach new topics in physical sciences, but in some cases the best way to illustrate a topic is in the context of a real-world example. In cases where a little science is required, it will always be explained. *The assessment materials will test only your knowledge and understanding of the mathematics.*

To illustrate the relevance of the mathematics to the real world, some additional material will be included in brown boxes such as the one below. We recommend that you read the material in these boxes, but do not get stuck or discouraged if you do not understand everything. All of the material in this style of boxed text is non-examinable, but it is hoped that you will enjoy reading about the applications and will want to learn more.

Note that green boxes contain essential information that *must* be read and understood.

The 'unreasonable effectiveness of mathematics'

As an illustration of why studying the mathematics underlying physical sciences can be so rewarding, consider Figure 1. The Cartwheel galaxy collided with another galaxy – presumably the small blue object to the left of the image – roughly 200 million years before the image that we now see.

Unleashing awesome energy, the collision sent a shock wave into the sparse gas around it. Expanding at nearly 100 kilometres per second, this 'cosmic tsunami' ploughed up a concentration of hydrogen gas and dust, creating conditions favourable for the birth of new stars around the galaxy's rim. The new stars in the rim are massive and extremely bright. Many have lived fast, died young and exploded,

Figure 1 The Cartwheel galaxy (a composite image in false colour)

leaving behind burnt-out cores that are 5×10^{13} times denser than iron. Others have formed black holes that swallow up surrounding matter; as matter falls into a black hole, never to be seen again, it can produce intense bursts of X-rays. All these processes of galaxy collision, shock wave creation, star birth and death, and the creation of X-rays are explained in terms of mathematical models covering situations that are far from familiar everyday experience, but nevertheless trusted by physicists and astronomers. It is a remarkable testament to the power of mathematics, and our understanding of physical laws, that this is possible.

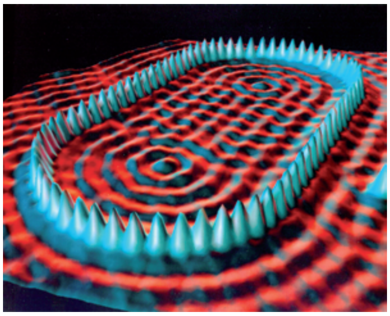


Figure 2 Iron atoms forming a 'quantum corral' (an image in false colour produced by a scanning tunnelling microscope)

The ring in the image of the Cartwheel galaxy is about 1.4×10^{21} metres in diameter. The image in Figure 2 reveals the equally strange world of the very tiny. It shows something called a *quantum corral* imaged by an immensely powerful microscope (a so-called *scanning tunnelling microscope*). The lumps around the edge of the corral are individual iron atoms standing proud of a flat copper surface, and the diameter of the corral is about 10^{-8} metres. Of most interest to scientists are the ripples inside the corral. These show concentrations of electrons that form patterns similar to water waves inside a bucket or the vibration patterns on the surface a drum. These analogies are close, though not exact. The interesting thing, from our point of view, is that water waves, drum vibrations and the concentrations of electrons in a quantum corral can all be understood using similar mathematical techniques applied to slightly different equations. In the case of the electrons, we are completely beyond the comfortable familiarity of everyday life, but the tools of mathematics again provide the keys to understanding. The calculations are demanding and use sophisticated physical principles, but most of the mathematics is built on what you will study in this module.

It cannot always have been obvious that all this would be possible. The Nobel laureate Eugene Wigner has spoken about what he called 'the unreasonable effectiveness of mathematics in the Natural Sciences'. For example, he points out that Newton was able to verify his law of gravitation to within 4%, but the law turned out to be accurate to better than one part in a million: outrageous fortune or unerring instinct? He concludes that:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
>
> (Eugene Wigner, 1960)

Study guide

The module is divided into thirteen units, which you will study at a rate of roughly one per fortnight. Each unit will emphasise one major topic or technique. The units are grouped into four books, dealing with connected sets of topics.

The first book is primarily about differential equations, but this first unit is an exception. It reviews material that you will need in order to make a good start on MST224. The material is broad-ranging, so this unit contains more pages than others, but it may take no longer to study. This is because most of its topics have been covered in previous modules. If you find some things familiar, you may not need to study everything in great detail.

To help you to judge where to put your effort, most subsections begin with a diagnostic test. If you can answer the test questions correctly, it is probably safe to skim through some subsections, just checking that you remember the key ideas. If you choose not to study a subsection in detail, do check any new terms that are introduced, and make sure that you understand them. (New terms are set in bold type.)

If you do not have time to study the whole unit, make sure that you are familiar with the basic properties of the exponential and trigonometric functions (Sections 2 and 3), and with techniques for differentiation and integration (Sections 5 and 6). These topics are required in almost every part of the module, and without a reasonable knowledge of them you will get stuck later on.

The unit contains a number of 'standard formulas': for example, for the solution of a quadratic equation, for expanding $\sin(a+b)$ and $\cos(a+b)$, and for the derivatives and integrals of standard functions. It will be helpful if you are able to remember such formulas, but not essential; they are all given in the module Handbook. You *do* need to be aware that these formulas exist, know when they are needed and how they are used, and be able to find them quickly in the Handbook.

1 Some elementary functions

Functions play a central role in mathematics. After a brief look at some general ideas about functions (in Subsection 1.1), this section reviews some very simple but nevertheless important types of function: linear functions, quadratic functions and powers. It also looks briefly at how functions may be combined by *composition*.

1.1 Functions, variables and parameters

Diagnostic test

Try Exercise 1. If your answer agrees with the solution, you may proceed quickly to Subsection 1.2. If not, then it is advisable to read this subsection more thoroughly.

Consider the following example of defining a function. At midday on 1 June, a reservoir contains 2×10^6 cubic metres of water. For the next 50 days, the reservoir loses 15 cubic metres per minute. Suppose that at a time t minutes after midday on 1 June, the reservoir contains V cubic metres of water. Then we might use the equation

$$V = 2 \times 10^6 - 15t \quad (0 \leq t \leq 72\,000) \tag{1}$$

to model the volume of water in the reservoir for the 50 days ($= 72\,000$ minutes) after midday on 1 June. The letters V and t represent measurable quantities. V and t are called **variables**. Here, V depends on t, so V is called the **dependent variable** and t the **independent variable**. The comment in parentheses simply says that t lies between 0 and 72 000.

We sometimes say that the function 'maps' each input value to its corresponding output value.

A **function** is a *process* or *rule* that can be applied to each of a specified set of input values to produce a definite output value. One example is: 'given t between 0 and 72 000, calculate $2 \times 10^6 - 15t$'. If we denote this function by f, then we can write equation (1) as $V = f(t)$, where $f(t)$ is the result of applying the rule f to the input value t.

The **domain** of a function is the set of permitted input values. The function f associated with equation (1) has as domain the set of real numbers t with $0 \leq t \leq 72\,000$ – that is, the interval $[0, 72\,000]$. The **image set** of a function is the set of output values. The function f associated with equation (1) has as image set the set of values of $2 \times 10^6 - 15t$ with $0 \leq t \leq 72\,000$ – that is, the interval $[920\,000, 2\,000\,000]$. This is the range of volumes of water (measured in cubic metres) in the reservoir over the 50 days following midday on 1 June.

To define a function fully, we must give both the function rule and the domain. So the function corresponding to equation (1) is

$$f(t) = 2 \times 10^6 - 15t \quad (0 \leq t \leq 72\,000), \tag{2}$$

where the expression $2 \times 10^6 - 15t$ on the right-hand side gives the *rule* or *formula* that specifies the function, and the conditions in parentheses indicate the domain (the range of input values for which the function is valid).

To define a function, a process must produce a unique output value for each allowed input. So, for example,

$$f(x) = \pm\sqrt{x} \quad (x \geq 0)$$

does *not* define a function f because $\pm\sqrt{x}$ does *not* specify a *unique* value for a given x.

We write $\pm\sqrt{x}$ to denote the positive and negative square roots of x because, by convention, $\sqrt{x}$ denotes only the positive square root.

You may have noticed a subtle difference between equations (1) and (2). The former gives a relationship between the variables V and t, but the latter defines a function – the abstract rule underlying this relationship. In this module, however, functions will often be labelled in a way that blurs this distinction. For example, we may write $V = V(t)$ rather than introducing a separate symbol, such as f, for the function relating V to t. In this notation, $V(30)$ is the value of V when $t = 30$. This is the volume of water in the reservoir 30 minutes after midday on 1 June. But $V(t)$ is generally taken to represent the function itself. If there is any doubt, the context will make it clear whether we are referring to a function or just one of its values.

We can also consider a generalisation of the reservoir model. Suppose that the reservoir initially contains V_0 cubic metres of water and that the rate of loss is L cubic metres per minute. We will assume that this situation persists for 72 000 minutes after a starting time $t = 0$. Then we have

$$V = V(t) = V_0 - Lt \tag{3}$$

for $0 \leq t \leq 72\,000$. We now have an equation involving several symbols, with differing roles. Assuming that we want to use equation (3) to describe how V changes with time t, we continue to call t the independent variable and V the dependent variable. The function $V(t)$ tells us how to relate a time t to V. The quantities V_0 and L do *not* depend on t. They may, however, take different values in different uses of equation (3) – in an application to a different reservoir, for example. V_0 and L are called **parameters**. Whatever the values of the parameters V_0 and L, equation (3) gives a similar form of relationship between V and t: for example, the independent variable t appears in a similar way in any of the expressions $12\,000 - 5t$, $300 - 6.6t$ and $14 - 2t$.

Often, having defined a function that maps x to y (say), it is useful to define another function that reverses this operation, so that if $y = f(x)$, we might wish to define another function g such that $x = g(y)$. In this case g is called the **inverse function** of f. As an example, if $f(x) = x^2$ with domain $x \geq 0$, then solving $y = f(x)$ for x gives the inverse function $x = g(y) = \sqrt{y}$. By definition, the domain of the inverse function $g(y)$ is the image set of $f(x)$, and the image set $g(y)$ is the domain of $f(x)$.

We ignore the negative solution $-\sqrt{y}$ because we know $x \geq 0$.

Exercise 1

Consider the function

$$f(x) = 2x^2 + 6 \quad (x \geq 0).$$

(a) State the domain and image set of f.

(b) Find the inverse function, and state its domain and image set.

A note on units and dimensions

If you have taken a science course previously, you will be aware that most physical quantities have units or dimensions associated with them. This module will often use SI units: the SI units of distance and time are metres and seconds (denoted by m and s, respectively), and the SI units of volume are cubic metres (denoted by m^3). In equations where symbols represent physical quantities, there are two acceptable ways of combining units and symbols; both are correct, but they must not be mixed together.

Method A The units are incorporated into the symbols

For example, the volume of the reservoir is written as $V = 10^6\,\mathrm{m}^3$, where m denotes metres. In this convention, the symbol V has units of m^3.

The notation $\mathrm{m}^3\,\mathrm{s}^{-1}$ is shorthand for 'metres cubed per second'.

Then, if we use an equation like (3), with initial volume $V_0 = 10^6\,\mathrm{m}^3$, rate of water loss $L = 10\,\mathrm{m}^3\,\mathrm{s}^{-1}$ and time $t = 1000\,\mathrm{s}$, we write

$$\begin{aligned} V &= 10^6\,\mathrm{m}^3 - (10\,\mathrm{m}^3\,\mathrm{s}^{-1} \times 1000\,\mathrm{s}) \\ &= 10^6\,\mathrm{m}^3 - 10\,000\,\mathrm{m}^3 \\ &= 990\,000\,\mathrm{m}^3. \end{aligned}$$

Method B The symbols are pure numbers

For example, the volume of the reservoir is written as V cubic metres, where $V = 10^6$ is a number with no units. In this convention, the units are stated in the text, but the equations involve pure numbers.

In this approach, if we use an equation like (3) with initial volume $V_0 = 10^6$ (in cubic metres), rate of water loss $L = 10$ (in cubic metres per second) and time $t = 1000$ (in seconds), then we write

$$V = 10^6 - (10 \times 1000) = 990\,000.$$

It is then helpful to round off the calculation with a sentence that includes appropriate units, for example stating that the volume in the reservoir at time 1000 seconds is 990 000 cubic metres.

This module generally adopts the latter approach, so units will not appear in equations. If units are required in the final answer, the wording of the question will guide you as to what they should be.

1.2 Linear functions

Diagnostic test

Try Exercises 2(a) and 3. If your answers agree with the solutions, you may proceed quickly to Subsection 1.3.

A **linear function** relating y to x is one of the general form

$$y = y(x) = mx + c,$$

where m and c are constants. The graph of such a function is a straight line (hence the term 'linear'), as in Figure 3. The constant c represents the value of y at the point where the line crosses the y-axis. The **gradient** (or **slope**) of the graph is the same everywhere, and is equal to m. That is, for any two points (x_1, y_1) and (x_2, y_2) on the graph, we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In this module, if no domain is specified for a function, and there are no obvious points where it is not defined, you can assume that the domain is $\mathbb{R}$, i.e. the set of all real numbers.

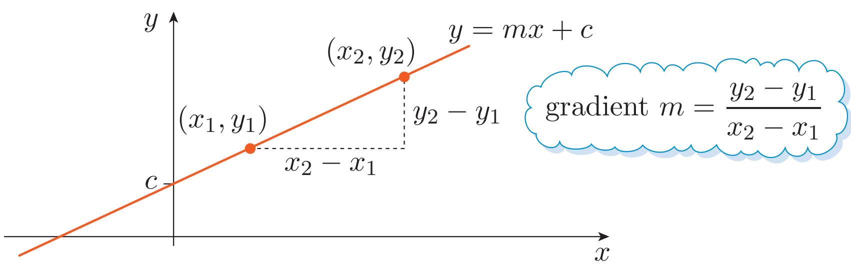


Figure 3 The graph of the linear function $y(x) = mx + c$

One situation where linear functions arise is when an object moves in a straight line with constant speed. Let us look at an example.

At 11.00 pm, a smuggler's boat passes a detector buoy that is 2 kilometres from a port. The boat then moves at a steady 5 metres per second on a straight course directly away from the port (along the line AZ in Figure 4). A coastguard vessel leaves the port in pursuit at midnight, travelling at 7 metres per second. When will it catch the smuggler's boat?

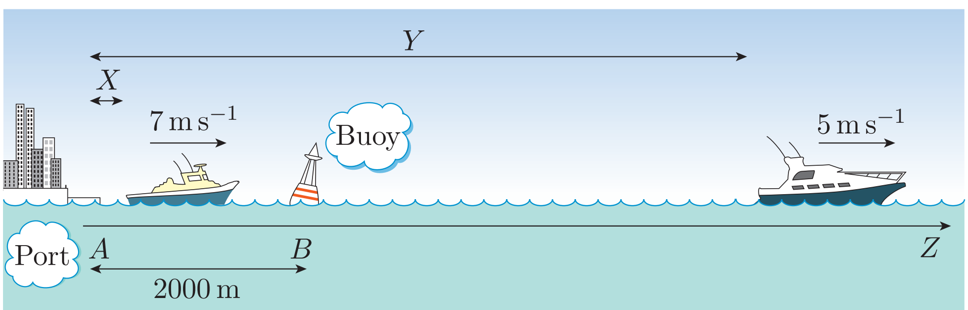


Figure 4 A boat chase modelled with linear functions

Suppose that we choose to measure time in seconds, starting from midnight, and distance in metres, measured from the port A. Let X metres be the distance of the coastguard vessel from A at time t seconds after midnight, and let Y metres be the distance of the smuggler's boat from A at the same time. We can readily obtain an expression for X in terms of t, since $X = 0$ when $t = 0$, and the coastguard vessel travels at a constant speed of 7 metres per second: we have

$$X = 7t.$$

We also want an expression giving Y in terms of t. The boat is moving at a constant speed of 5 metres per second, so Y will be related to t by a linear equation of the form

$$Y = 5t + c,$$

where c is a constant. We also know that (at point B) $Y = 2000$ at 11.00 pm, which is 1 hour, or 3600 seconds, before midnight and so corresponds to $t = -3600$.

Exercise 2

(a) Find the value of c such that $Y = 5t + c$ satisfies the condition $Y = 2000$ at $t = -3600$.

(b) (i) When will the coastguard vessel catch the smuggler's boat?

(ii) The limit of territorial waters is 100 kilometres from A. Will the vessel catch the boat within territorial waters?

Simultaneous linear equations

Surveying equipment often uses laser beams, which travel in straight lines and can therefore be described by linear equations. We might wish to locate a point in space by finding the position where two laser beams cross. This is one of a wide variety of situations where we need to find the intersection of two straight-line graphs, which is equivalent to the algebraic problem of solving two **simultaneous linear equations**. Consider, for example, the following linear equations (graphed in Figure 5):

These equations are linear, since they can be rewritten in the form $y = mx + c$.

$$4x + 3y = -1, \tag{4}$$

$$3x + y = 3. \tag{5}$$

There are many ways of solving these equations: one powerful method is **Gaussian elimination**. Let us see how this works for equations (4) and (5).

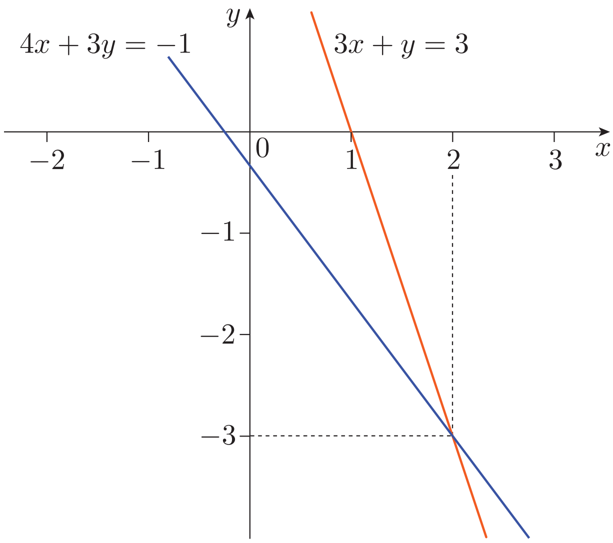


Figure 5 Graphical representation of the equations $4x + 3y = -1$ and $3x + y = 3$, and their solution

The aim of the method is to subtract a multiple of the first equation from the second in order to eliminate the x terms. First, we multiply equation (4) by $\frac{3}{4}$, to obtain an equation with the same coefficient of x as in equation (5):

$$\tfrac{3}{4} \times 4x + \tfrac{3}{4} \times 3y = \tfrac{3}{4} \times (-1),$$

which simplifies to

$$3x + \tfrac{9}{4}y = -\tfrac{3}{4}. \qquad (6)$$

Now we subtract equation (6) from equation (5). This eliminates x, and gives

$$y - \tfrac{9}{4}y = 3 - \left(-\tfrac{3}{4}\right) = 3 + \tfrac{3}{4} = \tfrac{15}{4},$$

that is, $-\frac{5}{4}y = \frac{15}{4}$, so $y = -3$.

To find x, we substitute this value of y into equation (4), to obtain

$$4x + 3(-3) = -1,$$

which gives $4x = -1 + 9 = 8$, and hence $x = 2$.

So the solution of equations (4) and (5) is $x = 2$, $y = -3$.

You may like to check these values by substituting them into equations (4) and (5).

Exercise 3

Use Gaussian elimination to solve the following equations for u and v:

$$2u - 5v = 19,$$
$$3u + 4v = -29.$$

1.3 Quadratic functions

Diagnostic test

Try Exercises 4(a) and 5. If your answers agree with the solutions, you may proceed quickly to Subsection 1.4.

A **quadratic function** relating y to x is a function of the general form

$$y = y(x) = ax^2 + bx + c, \tag{7}$$

where a, b and c are constants, and $a \neq 0$. The graph of such a quadratic function is a **parabola**, and may open 'up' or 'down', depending on the sign of a, as illustrated in Figure 6.

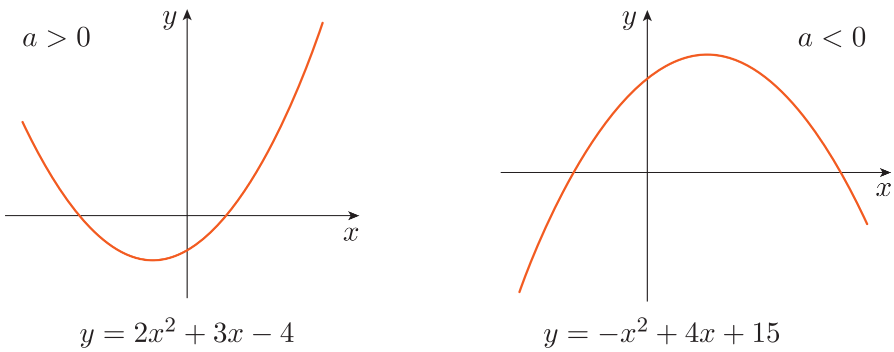


Figure 6 Graphical representation of two quadratic functions

If $a > 0$, then the graph opens 'up' and positive values of y may become arbitrarily large, but there is a smallest (minimum) value that y can take. If $a < 0$, then the graph opens 'down', and negative values of y may become arbitrarily large in magnitude, but there is a largest (maximum) value that y can take.

For example, suppose that a ball is thrown directly upwards at time $t = 0$ seconds, with initial velocity $10\,\mathrm{m\,s^{-1}}$, from a height of 2.0 metres (see Figure 7).

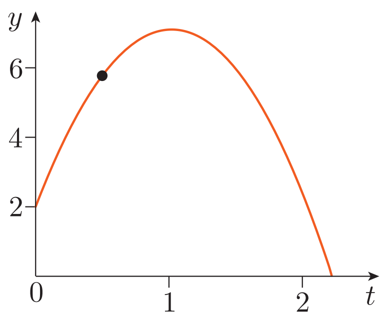


Figure 7 A ball is thrown vertically upwards

The ball moves under the influence of gravity. Its position, y metres above the ground after t seconds, is given by

$$y = -4.9t^2 + 10t + 2.$$

This equation may be taken on trust.

Suppose that we want to find when the ball will hit the ground – that is, the value of t when $y = 0$. Then we need to solve the **quadratic equation**

$$-4.9t^2 + 10t + 2 = 0. \tag{8}$$

You will have met the formula for the solution of a general quadratic equation before. It is given below.

Procedure 1 Solution of a quadratic equation

The quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b and c are constants, and $a \neq 0$, can be solved for x using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (9)$$

The solutions of a quadratic equation are often referred to as its **roots**.

Notice that the **sum of the roots** is $-b/a$, which is a useful check.

The term 'root' is also used for a solution of other sorts of equation, as you will see in Section 4.

Using this formula to solve equation (8) for t gives

$$t = \frac{-10 \pm \sqrt{100 + 39.2}}{-9.8} = 2.2 \text{ or } -0.18$$

(to two significant figures). Here the solution $t = -0.18$ refers to a time before the ball is thrown, so it can be discarded. The ball hits the ground about 2.2 seconds after it is thrown.

In this example, the quadratic equation has two solutions, but this is not always true. Look at the graphs in Figure 6, and imagine moving them up and down (which corresponds to varying the value of c). It is clear that the x-axis may meet a quadratic graph in two places, or it may touch at just one place (where the graph has a maximum or minimum value), or it may not meet the graph at all. Examples of these three cases are shown in Figure 8.

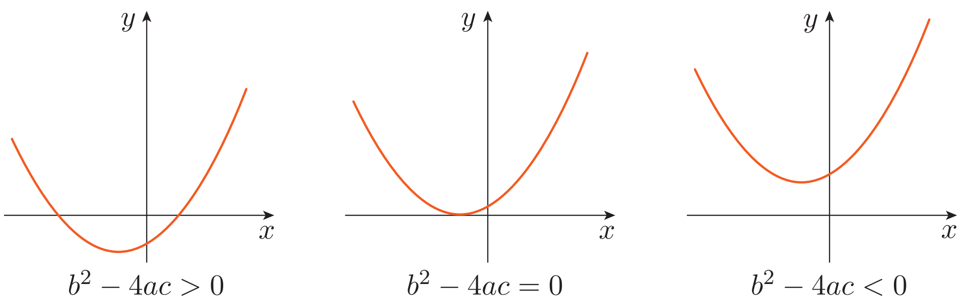


Figure 8 Graphs of the quadratic equation $y = ax^2 + bx + c$ for $a > 0$ and different values of $b^2 - 4ac$

The three cases can be understood by inspecting equation (9), which involves the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, we obtain a positive value for $\sqrt{b^2 - 4ac}$, leading to two distinct solutions.
- If $b^2 - 4ac = 0$, it follows that $\sqrt{b^2 - 4ac} = 0$, leading to just one solution. (Although, as you will see below, this one solution is sometimes regarded as two solutions with equal values.)
- If $b^2 - 4ac < 0$, there are no real values for $\sqrt{b^2 - 4ac}$, leading to no real solutions.

In Section 4, you will see that complex numbers enable us to write down square roots of negative numbers, and hence to produce (complex) solutions of a quadratic equation even when $b^2 - 4ac < 0$.

The quantity $b^2 - 4ac$ is often referred to as the **discriminant** of the quadratic equation because it *discriminates* between these three cases.

Exercise 4

Solve the following equations for x.

(a) $2x^2 + 7x - 4 = 0$ (b) $x^2 + x - 6 = 0$

Sometimes, you may find that you need to solve a quadratic equation where the coefficients are letters rather than numbers.

Exercise 5

ω is the Greek letter omega. The Greek alphabet is given in the Handbook.

If K and ω are constants with $K \geq \omega$, show that the solutions (for x) of

$$x^2 + 2Kx + \omega^2 = 0$$

are $x = -K \pm \sqrt{K^2 - \omega^2}$.

The solutions of a quadratic equation correspond to a **factorisation** of the corresponding quadratic function. For example, $x^2 + x - 6 = 0$ has solutions $x = 2$ and $x = -3$, and this corresponds to the factorisation

$$x^2 + x - 6 = (x - 2)(x + 3).$$

You may like to check this by multiplying out $(x - 2)(x + 3)$.

With experience, you may find that such factorisations provide a convenient way of solving some quadratic equations, but the formula in equation (9) provides a reliable method that can be used in all cases.

One point of caution: if you want to factorise a quadratic function, you can do this by first solving the equation (e.g. by using equation (9)), but you need to be careful to match the coefficient of x^2 in the factorisation with that in the original quadratic function. For example, $2x^2 + 7x - 4 = 0$ has solutions $x = \frac{1}{2}$ and $x = -4$, but to factorise $2x^2 + 7x - 4$ we write

$$2x^2 + 7x - 4 = 2\left(x - \tfrac{1}{2}\right)(x + 4) = (2x - 1)(x + 4),$$

where the 2 is needed to ensure that the coefficients of x^2 are the same on each side.

It is helpful to recognise some particular factorisations. Two useful ones are

$$x^2 + 2Ax + A^2 = (x + A)^2 \quad \text{and} \quad x^2 - 2Ax + A^2 = (x - A)^2.$$

So, for example, $x^2 - 6x + 9 = (x - 3)^2$. Such quadratics are called **perfect squares**. Perfect squares correspond to quadratic equations in

which the discriminant $b^2 - 4ac$ is equal to zero. (You may like to check this for yourself.) Thus equations in which the discriminant is zero can be written in the form $(x + A)(x + A) = 0$ or $(x - A)(x - A) = 0$, and these factorisations lead us sometimes to consider such equations as having two *equal* roots ($x = -A$ and $x = -A$, or $x = A$ and $x = A$) rather just one root.

Another useful factorisation is

$$x^2 - A^2 = (x + A)(x - A).$$

So, for example, $x^2 - 16 = (x + 4)(x - 4)$. Such a quadratic is called a **difference of two squares**.

You need to be particularly careful when solving a quadratic equation that involves the *same* letters as appear in the standard formula (9), but in a *different* way.

Example 1

Solve for x the equation

$$abx^2 - (a + b)x + 1 = 0,$$

where a and b are non-zero constants.

Solution

You need to keep a cool head here, because the letters in equation (9) are used in a different way in the given equation. In equation (9), we need

$$ab \text{ for } a, \quad -(a + b) \text{ for } b, \quad 1 \text{ for } c.$$

So we obtain the solutions

$$x = \frac{a + b \pm \sqrt{(a + b)^2 - 4ab}}{2ab}.$$

This expression gives the solutions, but it turns out to be possible to express them in a much simpler form. We have $(a + b)^2 = a^2 + 2ab + b^2$, so the discriminant can be written as

$$(a + b)^2 - 4ab = (a^2 + 2ab + b^2) - 4ab = a^2 - 2ab + b^2 = (a - b)^2.$$

So the solutions can be written in the alternative form

$$x = \frac{a + b \pm \sqrt{(a - b)^2}}{2ab} = \frac{a + b \pm (a - b)}{2ab}.$$

Now $(a + b) + (a - b) = 2a$ and $(a + b) - (a - b) = 2b$, so the two solutions are $x = 1/a$ and $x = 1/b$.

This solution also follows from the factorisation $(ax - 1)(bx - 1) = 0$.

Exercise 6

Factorise the following expressions, where $a > 0$.

(a) $x^2 - a$ (b) $2x^2 - 8a$ (c) $x^4 - 6x^2 + 9$

1.4 Powers

Diagnostic test

Try Exercise 7. If your answer agrees with the solution, you may proceed quickly to Subsection 1.5.

In a^n, a is called the **base**, and n may be referred to as the **power**, the **index** or the **exponent**.

You will know that $10^5 = 10 \times 10 \times 10 \times 10 \times 10$. In general, a^n means the product of n copies of a (for any real number a and any positive integer n). In particular, $a^1 = a$.

For positive integers m and n, we have the property

$$a^n \times a^m = a^{n+m}, \qquad (10)$$

since each side is the product of $n+m$ copies of a. For example,

$$10^2 \times 10^5 = 10^7.$$

Consequently, if we multiply m copies of a^n, we obtain

$$\underbrace{a^n \times a^n \times a^n \times \cdots \times a^n}_{m \text{ times}} = a^{\overbrace{n+n+n+\cdots+n}^{m \text{ times}}};$$

that is,

$$(a^n)^m = a^{n \times m} = a^{nm}. \qquad (11)$$

For example, $(10^2)^3 = 10^6$.

Recall that the ***n*th root** of a number a is a number b such that $b^n = a$, and we write $b = \sqrt[n]{a}$.

The definition of a^n can be extended to cases where n is not a positive integer by assuming that equations (10) and (11) hold more generally. For $a \neq 0$, this assumption leads to the definition of a^0 as 1, and a^{-n} as $1/a^n$; and, for $a > 0$, to the definition of $a^{1/n}$ as the nth root of a, and $a^{m/n}$ as the nth root of a^m. So, for example:

$$10^{-4} = 1/10^4 = 0.0001;$$

$$27^{1/3} = \sqrt[3]{27} = 3 \quad (\text{since } 3^3 = 27);$$

$$4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{\sqrt[2]{4^3}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

The negative square root of 5, for example, would be written as $-5^{1/2}$ or $-\sqrt{5}$.

If a is positive, it is conventional to take the fractional power of a to be positive. So, for example, $9^{1/2} = \sqrt{9}$ means 3 rather than -3. If a is negative, fractional powers of a do not necessarily exist – at least not as real numbers. (The square roots of negative numbers are discussed in Section 4.)

The following properties of powers hold for all real numbers $a > 0$ and all real exponents x and y. These properties are not proved here, but we will make use of them as necessary.

$$a^x > 0, \qquad (12)$$

$$a^{-x} = 1/a^x, \qquad (13)$$

$$a^{x+y} = a^x \times a^y, \tag{14}$$
$$a^x/a^y = a^{x-y}, \tag{15}$$
$$(a^x)^y = a^{x\times y} = a^{xy}. \tag{16}$$

We also note that $0^x = 0$ for all $x \neq 0$.

Finally, given *positive* numbers a and b, we have the following rules for products and quotients:

$$(ab)^x = a^x \times b^x \quad \text{and} \quad (a/b)^x = a^x/b^x.$$

For example, $15^7 = 3^7 \times 5^7$ and $(5/3)^4 = 5^4/3^4$.

These rules do *not* apply if a or b is negative. For example, $\sqrt{(-1)\times(-1)} \neq \sqrt{(-1)}\sqrt{(-1)}$.

Exercise 7

Use the properties of indices to simplify each of the following, where $a > 0$ and x is real.

(a) a^3a^5 (b) a^3/a^5 (c) $(a^3)^5$ (d) $(2^{-1})^4 \times 4^3$

(e) $8^{-1/3}$ (f) $16^{3/4}$ (g) $\left(\frac{4}{9}\right)^{3/2}$ (h) $(16x^4)^{1/2}$

1.5 Combining functions

Diagnostic test

Try Exercise 8. If your answer agrees with the solution, you may proceed quickly to Section 2.

Consider the following, which leads to an example of the *composition* of two functions. A hiker travels a distance of x kilometres after walking for a time t hours along a path. As he travels, the path rises up a mountain, so that after walking x kilometres along the path, his height (in kilometres) is h.

We might express x as a function of t using a function f, so that $x = f(t)$. As a practical example, the distance might be

$$x = f(t) = 2t - \tfrac{1}{4}t^2 \quad (\text{for } 0 < t < 2).$$

Similarly, the height can be expressed as a function of x by another function g, so that $h = g(x)$. For example, the height might be

$$h = g(x) = \tfrac{1}{2}x - \tfrac{1}{8}x^2 \quad (\text{for } 0 < x < 3).$$

Now, we might want to determine the hiker's height h as a function of time t. With $h = \frac{1}{2}x - \frac{1}{8}x^2$ and $x = 2t - \frac{1}{4}t^2$, we have

$$\begin{aligned} h = \tfrac{1}{2}\left(2t - \tfrac{1}{4}t^2\right) - \tfrac{1}{8}\left(2t - \tfrac{1}{4}t^2\right)^2 &= \left(t - \tfrac{1}{8}t^2\right) - \tfrac{1}{8}\left(4t^2 - t^3 + \tfrac{1}{16}t^4\right) \\ &= t - \tfrac{5}{8}t^2 + \tfrac{1}{8}t^3 - \tfrac{1}{128}t^4, \end{aligned}$$

so we can write $h = H(t)$, where $H(t)$ is the function

$$H(t) = t - \tfrac{5}{8}t^2 + \tfrac{1}{8}t^3 - \tfrac{1}{128}t^4.$$

It is a very common procedure to express a quantity that is a function of one variable (height as a function of position, in our example) in terms of another variable (in this case, time). There is a shorthand way of describing what is being done in mathematical language. The relation between the function H and the functions f and g is written

$$H(t) = g(f(t)),$$

meaning that to obtain the function H we apply the function f to t, and then the function g to the result. We say that H is the **composition** of the function g with the function f.

In the hiker example above, the function $g(x)$ gives height as a function of position, while the function $H(t)$ gives height as a function of time. These are different functions, and texts in pure mathematics are usually strict about this distinction, and use different symbols for these functions, just as we have done above. When mathematics is used in the physical sciences, however, another approach is frequently used: we write $h(x)$ to denote height as a function of distance, and $h(t)$ to denote height as a function of time. The symbol inside the brackets (x or t) indicates which of the functions $h(x) = \frac{1}{2}x - \frac{1}{8}x^2$ or $h(t) = t - \frac{5}{8}t^2 + \frac{1}{8}t^3 - \frac{1}{128}t^4$ is intended. If this approach were not adopted, we would have to introduce a different symbol for a quantity every time we make a change of variable, which would make some physics texts impossible to read. The approach works well so long as you do not use $h(x)$ and $h(t)$ in the same equation.

Later on, when doing calculus, you will see that it is useful to be able to recognise a complicated function as the composition of simpler ones. Remember that the composite function $g(f(x))$ tells us to 'apply f first, then g', while $f(g(x))$ tells us to 'apply g first, then f'. The 'inner' function is applied before the 'outer' function, and the order in which the functions are applied matters!

Example 2

Express the function

$$h(x) = \frac{1}{\left(\sqrt{1+2x^2}\,\right)^3} \tag{17}$$

as a composite of a quadratic function and a power function.

Solution

Note first that $1 + 2x^2$ is quadratic and that by writing $y = 1 + 2x^2$, the right-hand side of equation (17) becomes $\dfrac{1}{(\sqrt{y})^3} = y^{-3/2}$ (a power).

So we can obtain $h(x)$ in two steps.

Step 1 Calculate $y = 1 + 2x^2$.

Step 2 Apply $\dfrac{1}{(\sqrt{y})^3} = y^{-3/2}$ to the result of Step 1.

Here, the domain of g is $x > 0$, but since $f(x) = 1 + 2x^2$ is always greater than 0, there is no problem.

So if $f(x) = 1 + 2x^2$ and $g(y) = y^{-3/2}$, then $h(x) = g(f(x))$.

Exercise 8

(a) If $f(x) = x^2$ and $g(x) = 1/(x-1)$, with $x > 1$, find the following.

(i) $f(g(x))$ (ii) $g(f(x))$

(b) Express $h(x) = \sin(1 + x^2)$ as a composition of three basic functions.

2 The exponential and logarithm functions

Diagnostic test

Try Exercise 11. If your answer agrees with the solution, you may proceed quickly to Section 3.

This diagnostic test covers both the subsections in this section.

2.1 The exponential function

The exponential function plays a central role in mathematics. Here its definition is given, followed by two of its most significant properties, before we examine some of their consequences.

The exponential function $\exp(x)$ can be defined by writing an infinite series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \tag{18}$$

You may have met other definitions of $\exp(x)$. These can be shown to be equivalent to equation (18).

where $n!$ is the factorial of n, that is,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

For example, $2! = 2$, $3! = 6$ and $4! = 24$.

This definition involves adding an infinite number of terms together. In general, dealing with infinite series can be a source of difficulties; for example, there will be problems if the terms keep growing as you go to larger and larger values of n. There is no problem with the series for the exponential function: as you add more terms in equation (18), their sum eventually approaches a definite number $\exp(x)$, which depends on x. The infinite series in equation (18) is called the **Taylor series** for $\exp(x)$.
A graph of the exponential function is shown in Figure 9.

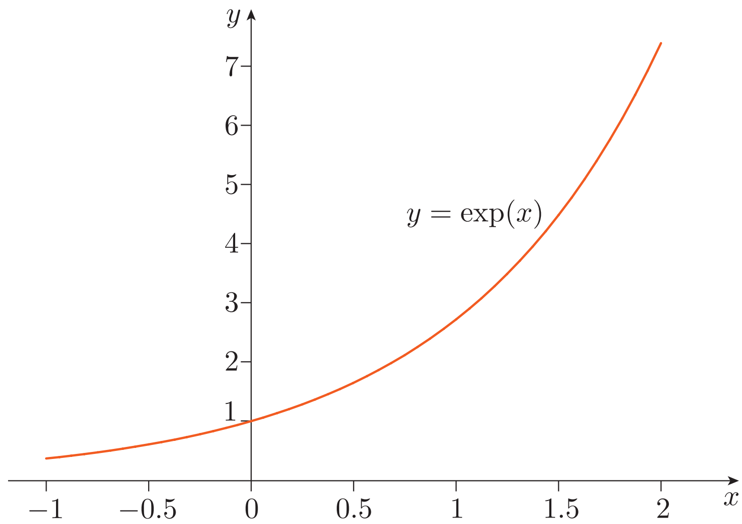


Figure 9 Graph of the exponential function

Exercise 9

With the help of a calculator, add together the first six terms in equation (18) for $x = 1/2$. Repeat your calculation for $x = 1$ using seven terms, and for $x = 2$ using eight terms. Hence obtain estimates for $\exp(1/2)$, $\exp(1)$ and $\exp(2)$.

Now for the two important properties of the exponential function. The first is that

$$\frac{d\exp(x)}{dx} = \exp(x). \tag{19}$$

Differentiation is discussed in Section 5.

That is, when you differentiate the exponential function, the result is just the function you started from.

The second property is that

$$\exp(x + y) = \exp(x)\exp(y). \tag{20}$$

These two properties (equations (19) and (20)) touch on so many aspects of mathematics that their power will become clear only as you progress through the module.

Equation (19) makes the exponential function a powerful tool for dealing with differential equations, as you will see later in this book. In order to see that this expression is correct, apply the rule for differentiating x^n to each term in the definition, equation (18). This gives

Note that $\frac{n}{n!} = \frac{1}{(n-1)!}$.

$$\begin{aligned}\frac{d}{dx}\exp(x) &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \\ &= \exp(x).\end{aligned}$$

Equation (20) is more difficult to derive directly from the definition, but the following exercise checks that it is correct in particular cases.

Exercise 10

Use the numerical values of $\exp(1/2)$, $\exp(1)$ and $\exp(2)$ quoted in the solution to Exercise 9 to provide a numerical check that $\exp(1/2) \times \exp(1/2) = \exp(1)$ and $\exp(1) \times \exp(1) = \exp(2)$, as predicted by equation (20).

An alternative notation

The property $\exp(x + y) = \exp(x)\exp(y)$ is analogous to the property noted earlier for powers:

$$a^{x+y} = a^x a^y \quad (\text{for } a > 0).$$

This suggests that it is sensible to regard $\exp(x)$ as the power of some positive number. We write

$$\exp(x) = e^x, \tag{21}$$

where e is known as **Euler's number**. The value of e is easily found by putting $x = 1$ in equation (21). This gives

$$e = \exp(1) = 2.718\,281\,828\ldots,$$

This value is correct to only ten significant figures; in fact, e is an irrational number, with a never-ending and never-repeating decimal representation.

using a numerical result from Exercise 9. We then have $\exp(2) = e^2$, $\exp(3) = e^3$, and so on.

Writing the exponential function in the form $\exp(x) = e^x$, equation (20) takes the form

$$e^{x+y} = e^x e^y$$

for any numbers x and y, and this is a special case of the equation $a^{x+y} = a^x a^y$ for powers. The exponential function shares other properties with powers of positive numbers. In particular,

$$e^x > 0, \tag{22}$$

$$e^{-x} = \frac{1}{e^x}, \tag{23}$$

$$\frac{e^x}{e^y} = e^{x-y}, \tag{24}$$

$$(e^x)^y = e^{xy}. \tag{25}$$

Figure 10 Leonhard Euler (1707–1783)

The number e

The number e is named after Leonhard Euler (Figure 10), who explored many properties of the exponential function, but the discovery of the number e is credited to Jacob Bernoulli (Figure 11), who used it to solve a problem involving the repayment of interest. (Euler and Bernoulli are pronounced as 'oiler' and 'ber-noo-lee'.)

Figure 11 Jacob Bernoulli (1654–1705)

2.2 The natural logarithm

It is useful to have a function that does the opposite of the exponential function. This function is called the **natural logarithm** function, and is given the symbol $\ln(x)$. The natural logarithm is the inverse function of the exponential function, so if $y = \exp(x)$, then $x = \ln(y)$.

When x is a single symbol, we may also write $\ln x$ rather than $\ln(x)$.

Some texts give other notations for $\ln(x)$, such as $\log_e(x)$ and $\log(x)$. We often refer to $\ln(x)$ as *the* **logarithm** function (although other less useful functions are also logarithms!). 'Take the logarithm of x', for example, generally means 'apply the natural logarithm function to x, to give $\ln(x)$'.

Because the exponential of a real number is always positive (see Figure 9), the natural logarithm function $\ln(x)$ is defined only for $x > 0$. Its domain is the set of real positive numbers, $x > 0$. A graph of the natural logarithm function is shown in Figure 12.

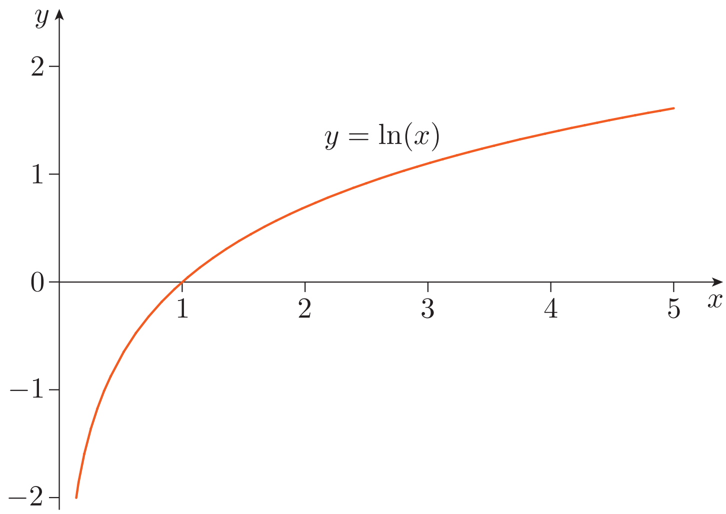


Figure 12 Graph of the natural logarithm function

The logarithm function obeys the important property

$$\ln(XY) = \ln X + \ln Y \tag{26}$$

for all $X > 0$ and $Y > 0$. To see why this is true, let $X = \exp(x)$ and $Y = \exp(y)$. Then, substituting in the left-hand side of equation (26) and using equation (20) gives

$$\ln(XY) = \ln(\exp(x)\exp(y)) = \ln(\exp(x + y)).$$

Because the logarithm and exponential are inverse functions of one another, the right-hand side becomes simply $x + y$. By replacing $x = \ln(X)$ and $y = \ln(Y)$, we obtain the fundamental property of the logarithm function given in equation (26).

Log tables and slide rules

Before electronic calculators became inexpensive, equation (26) was used to multiply or divide numbers. One converted numbers to their logarithms using a book of tables (log tables), added them (which is easier than multiplying), and used another table to find the answer.

An application of this idea is the slide rule, a mechanical calculator marked with logarithmic scales, where the addition step is performed by moving one scale past the other. Pilot training still requires knowledge of this device, partly because it still works when every electrical circuit has failed!

Common log tables (e.g. Figure 13) were based on another logarithm function, $\log_{10}$, defined so that if $10^y = x$, then $y = \log_{10} x$. This function is rarely used in advanced mathematics.

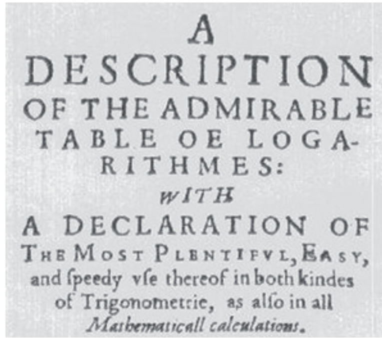

A
DESCRIPTION
OF THE ADMIRABLE
TABLE OE LOGA-
RITHMES:
WITH
A DECLARATION OF
THE MOST PLENTIFVL, EASY,
and ſpeedy vſe thereof in both kindes
of Trigonometrie, as alſo in all
Mathematicall calculations.

Figure 13 Part of the title page of a 1616 translation of John Napier's *A Description of the Admirable Table of Logarithmes*.

Several properties of the exponential function were listed in equations (22)–(25). By taking logarithms on both sides of these equations, we obtain the following results for the natural logarithm function:

$$\ln(1/u) = -\ln u, \tag{27}$$

$$\ln(u/v) = \ln u - \ln v, \tag{28}$$

$$\ln(u^v) = v \ln u. \tag{29}$$

Exercise 11

Simplify each of the following (where a, b and x are positive, and x and y are real).

(a) $\ln 7 + \ln 4 - \ln 14$ (b) $\ln a + 2\ln b - \ln(a^2 b)$

(c) $\ln(e^x \times e^y)$ (d) $e^{2\ln x}$ (e) $e^{-2\ln x}$

(f) $\exp(2\ln x + \ln(x+1))$

We sometimes want to calculate the value of

$$y = a^x$$

where $a > 0$ and x is not an integer or a simple fraction. For example, we may need to evaluate 5^π. The exponential and natural logarithm functions allow us to do this. We write

$$a = \exp(\ln(a)) = e^{\ln a},$$

and then

$$a^x = \left(e^{\ln a}\right)^x = e^{x\ln a}, \tag{30}$$

This calculation of a^x is built into most modern calculators.

which can be evaluated using the ln and exp functions.

Exercise 12

Use equation (30) to determine the values of $A = 5^\pi$ and $B = 10^{-4.315}$ to four significant figures.

Distinguish carefully between a function of the form x^a, and a function of the form a^x, where a is a constant. A function of the form x^a is called a **power function**. Examples include x^2, $x^{1/2}$ and $x^{-5/2}$. By contrast, a function of the form a^x, where $a > 0$, can be written as $a^x = e^{kx}$, where $k = \ln a$. We say that this describes an **exponential dependence**, although e^{kx} is only the *exponential function*, e^x, if $k = 1$.

3 Trigonometric functions

This section adds another class of functions to the 'library' developed in Sections 1 and 2. These are the trigonometric functions. They originate in the geometry of right-angled triangles, but in this module we are equally often concerned with their use in describing repetitive or oscillatory behaviour. In particular, they arise as solutions of certain differential equations.

Differential equations are equations involving derivatives. They are the subject of the next two units.

3.1 Introducing the trigonometric functions

Diagnostic test

Try Exercise 13. If your answer agrees with the solution, you may proceed quickly to Subsection 3.2.

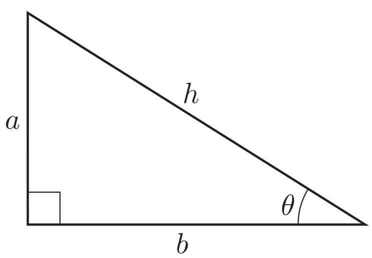


Figure 14 Trigonometric ratios: $\sin\theta = a/h$, $\cos\theta = b/h$, $\tan\theta = a/b$

You will have met $\sin\theta = a/h$, $\cos\theta = b/h$ and $\tan\theta = a/b$ as ratios in a right-angled triangle (see Figure 14). However, these definitions of the sine, cosine and tangent functions work only for $0 < \theta < \pi/2$, where θ is in radians.

Recall that $180° = \pi$ radians.

Note that many of the formulas in this module are valid only in radians, and angles will almost always be expressed in radians.

To define the **sine** and **cosine** functions for a general value of θ, we can use Figure 15, which shows a circle of radius 1. Imagine that the line OA started along the x-axis, and was then rotated anticlockwise through an angle θ. Then the point A has coordinates $(\cos\theta, \sin\theta)$. Here θ may have *any* value, and this *defines* the cosine and sine functions for all values of θ (positive, zero or negative). A negative value of θ corresponds to a rotation *clockwise*.

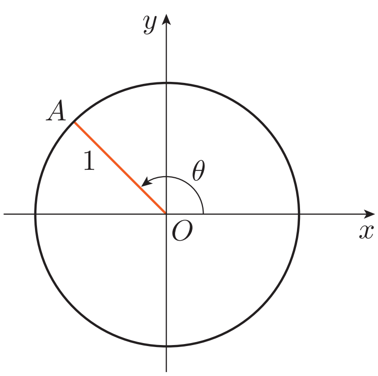


Figure 15 The point A has coordinates $(\cos\theta, \sin\theta)$

If we rotate through 2π radians (360°), then we go round a full circle. So rotations of θ and $\theta + 2\pi$ leave A in exactly the same place. This leads to the repetitive nature of the graphs of sin and cos: we have

$$\sin(\theta + 2\pi) = \sin\theta \quad \text{and} \quad \cos(\theta + 2\pi) = \cos\theta, \quad \text{for any } \theta$$

(see Figure 16). Functions like this, which repeat their values every 2π, are said to be **periodic**, with period 2π.

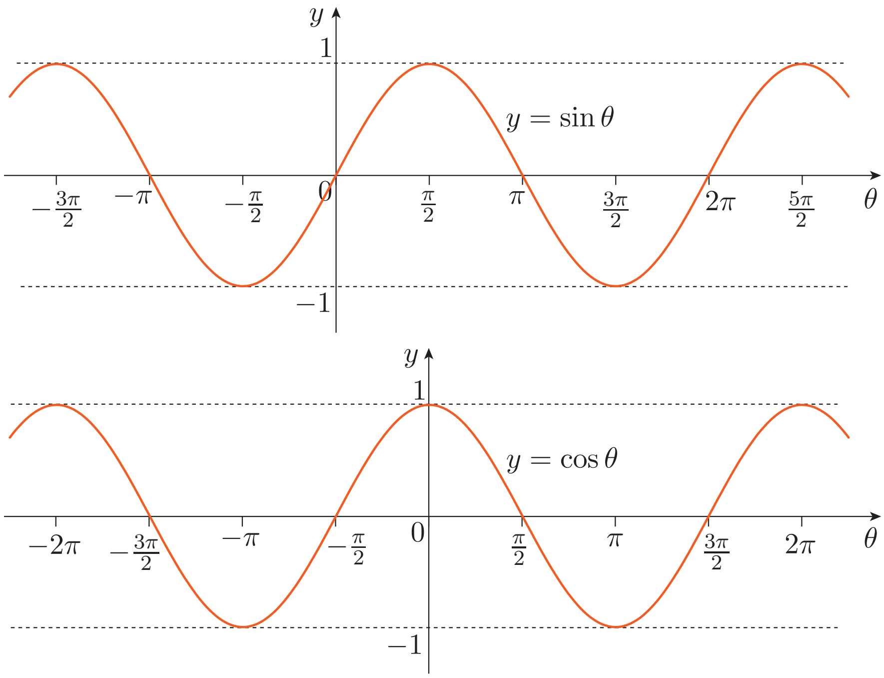


Figure 16 Graphs of the sine and cosine functions

You should also note that the graph of the cosine function is the same as that of the sine function, shifted to the left by $\pi/2$. We can express this mathematically as

$$\cos\theta = \sin(\theta + \pi/2).$$

Other trigonometric functions can be defined in terms of sin and cos. You will have met the **tangent** function $\tan\theta = \sin\theta/\cos\theta$. This is defined for all real θ except where $\cos\theta = 0$ (i.e. at $\theta = \pm\pi/2$, $\pm 3\pi/2$, and so on). You may also have met

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These functions are referred to as **secant**, **cosecant** and **cotangent**.

We need to restrict the domains of cosec and cot to exclude points where $\sin\theta = 0$, and the domains of sec and tan to exclude points where $\cos\theta = 0$.

Exercise 13

(a) Use Figure 15 to find the values of $\sin\theta$ and $\cos\theta$ for $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(b) Find the values of $\tan\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{2}$ in all cases where these values exist.

(c) Two right-angled triangles are shown in Figure 17. Use these to calculate the values of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\cot\theta$ for θ equal to each of $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$.

(d) For what values of θ is $\sin\theta = 0$? (Refer to Figure 16.)

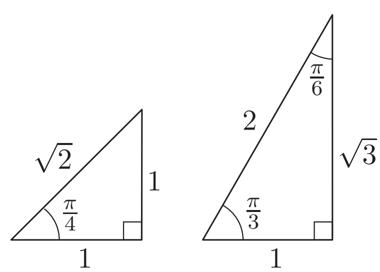


Figure 17 Two right-angled triangles

The graphs of sec, cosec and cot are given in the Handbook (as well as those of sin, cos and tan).

The function tan has the graph shown in Figure 18. Notice that $\tan\theta$ actually repeats its values every π. (This is because $\sin(\theta+\pi) = -\sin\theta$ and $\cos(\theta+\pi) = -\cos\theta$, so that $\tan(\theta+\pi) = \tan\theta$.)

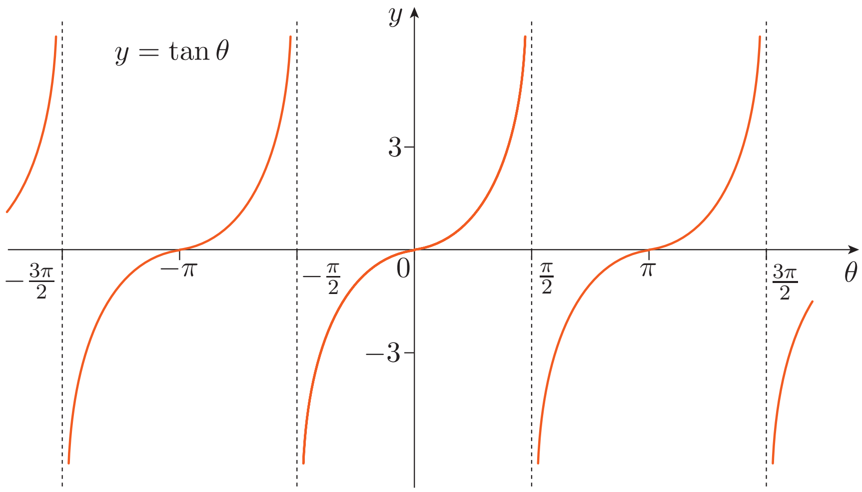


Figure 18 Graph of $y = \tan\theta$

3.2 Inverse trigonometric functions

Diagnostic test

Try Exercise 14. If your answer agrees with the solution, you may proceed quickly to Subsection 3.3.

Suppose that you need to solve for x the equation

$$\cos x = \tfrac{1}{2}.$$

What solutions are there? You have seen (in Exercise 13(c)) that $\cos\frac{\pi}{3} = \frac{1}{2}$, so one solution is certainly $x = \frac{\pi}{3}$. There are others, however. For instance, since cos repeats its values every 2π, another solution is $x = \frac{\pi}{3} + 2\pi$. We can find an infinite number of solutions by adding or subtracting multiples of 2π to/from $\frac{\pi}{3}$. There are even more solutions. If you look at the graph of cos in Figure 16, you can see that a horizontal line at $y = \frac{1}{2}$ would cut it twice between 0 and 2π: we also have $\cos\frac{5\pi}{3} = \frac{1}{2}$. And more solutions can be found by adding or subtracting multiples of 2π to/from $\frac{5\pi}{3}$.

In general, an equation of the form

$$\cos x = y \tag{31}$$

is solved for x by finding a value of the **inverse trigonometric function** arccos:

$$x = \arccos y.$$

However, we need to be careful here. Solutions of equation (31) are not unique, as we saw for $y = \frac{1}{2}$. If we reverse the roles of the axes for the cosine curve in Figure 16, we obtain the curve shown in Figure 19. However, this is *not* the graph of a *function*: a vertical line may meet the curve in many places, reflecting the fact that for a given y, equation (31) may have multiple solutions x. To ensure that $\arccos y$ has a unique value, we need to restrict the range in which values of arccos can lie. The *restricted values* of arccos are given in Table 1, together with those of two other inverse trigonometric functions, arcsin and arctan. In Figure 19, when the values taken by arccos are restricted, we obtain just the part of the curve shown in blue, which *is* a valid graph for a function.

Some texts use $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ rather than arcsin, arccos and arctan.

Graphs of arcsin and arctan, as well as arccos, are given in the Handbook.

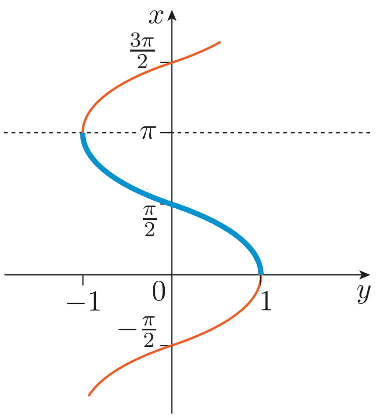


Figure 19 The graph of $x = \arccos y$ is just the blue part of the curve

Table 1 Domains and values of inverse trigonometric functions

Function	Inverse function	Domain of inverse function	Restricted values of inverse function
$y = \sin x$	$x = \arcsin y$	$-1 \leq y \leq 1$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
$y = \cos x$	$x = \arccos y$	$-1 \leq y \leq 1$	$0 \leq x \leq \pi$
$y = \tan x$	$x = \arctan y$	$\mathbb{R}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Calculators and computer software can be expected to give values of the inverse trigonometric functions drawn from suitably restricted values, usually those in Table 1. However, these values are not always appropriate in particular real-world problems. It is therefore important to be alert to the fact that an equation such as $\cos x = y$ actually has infinitely many solutions: if $|y| < 1$, there are two solutions in the range 0 to 2π, together with infinitely many others obtained by shifting these two by multiples of 2π.

For $|y| > 1$, $\cos x = y$ has no solutions.

Exercise 14

(a) Find all the solutions of $\sin\theta = 0.8$ in the range 0 to 6π.

(b) Find all the solutions of $\tan\theta = 1$.

You will need a calculator in order to get started.

3.3 Some useful trigonometric identities

Diagnostic test

Try Exercises 15(c), 16(a), 16(b) and 16(g). If your answers agree with the solutions, you may proceed quickly to Section 4.

Figure 20 shows the relation between a clockwise rotation through θ (regarded as a rotation through $-\theta$) and an anticlockwise rotation through θ. Notice that such rotations lead to equal x-coordinates but to y-coordinates of opposite signs. So we have

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta.$$

These relations hold for all values of θ, and are examples of **trigonometric identities**. Such identities can be useful in a variety of contexts, such as simplifying expressions involving trigonometric functions or evaluating integrals.

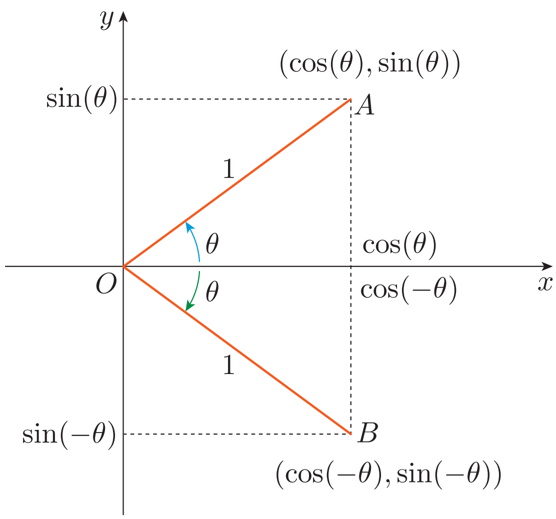


Figure 20 The effects of positive (anticlockwise) and negative (clockwise) rotations

There are three particularly useful trigonometric identities that you should remember.

$$\cos^2\theta + \sin^2\theta = 1, \tag{32}$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi, \tag{33}$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi. \tag{34}$$

Other useful trigonometric identities are given below. You should be aware of these identities, but there is no need to remember them, as they can all be very easily derived from equations (32)–(34) in one or two lines (see Exercise 15).

$$1 + \tan^2\theta = \sec^2\theta, \tag{35}$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta; \tag{36}$$

$$\tan(\theta \pm \phi) = (\tan\theta \pm \tan\phi)/(1 \mp \tan\theta\tan\phi)); \tag{37}$$

$$\sin\theta\sin\phi = \tfrac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)], \tag{38}$$
$$\cos\theta\cos\phi = \tfrac{1}{2}[\cos(\theta-\phi) + \cos(\theta+\phi)], \tag{39}$$
$$\sin\theta\cos\phi = \tfrac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]; \tag{40}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta, \tag{41}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta, \tag{42}$$
$$\tan(2\theta) = (2\tan\theta)/(1-\tan^2\theta). \tag{43}$$

All the trigonometric identities discussed in this subsection are included in the module Handbook for easy reference.

Exercise 15

Use equations (32)–(34) to derive the following identities.

(a) Equation (35) (b) Equation (37)

(c) Equation (38) (d) Equation (41)

Exercise 16

Use equations (33) and (34), and particular values of sin and cos, to simplify each of the following.

(a) $\sin(2\pi - \theta)$ (b) $\sin\left(\frac{\pi}{2} - \theta\right)$ (c) $\sin(\pi - \theta)$

(d) $\cos(\pi - \theta)$ (e) $\cos(2\pi - \theta)$ (f) $\cos\left(\frac{\pi}{2} - \theta\right)$ (g) $\cos\left(\frac{3\pi}{2} + x\right)$

Taylor series for trigonometric functions

The sine and cosine functions can be expressed as infinite series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots, \tag{44}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots. \tag{45}$$

Note that these equations are valid only in radians.

Because the factorial $n!$ becomes very large as n increases, these series sum to a finite number, no matter how large x is. They are the **Taylor series for the sine and cosine functions**. A frequent use of these expressions is to approximate $\sin\theta$ and $\cos\theta$ when θ is small.

$$\sin\theta \simeq \theta \quad (\text{for } \theta \ll 1 \text{ in radians}), \tag{46}$$
$$\cos\theta \simeq 1 - \tfrac{1}{2}\theta^2 \quad (\text{for } \theta \ll 1 \text{ in radians}). \tag{47}$$

The symbols $\ll$ and $\gg$ mean 'much smaller than' and 'much larger than', respectively.

4 Complex numbers

Diagnostic test

Try Exercises 17(a), 17(b), 17(e) and 18. If your answers agree with the solutions, you may proceed quickly to Subsection 4.2.

Complex numbers provide a system within which we can solve any quadratic equation (and, indeed, any polynomial equation). They are used in many of the mathematical techniques introduced in this module.

There is no *real* number x satisfying the equation

$$x^2 = -1.$$

However, there are circumstances where it is convenient to have a system of 'numbers' in which such an equation can be solved. Such a system is provided by the complex numbers. A **complex number** is one of the form

Engineers commonly use j to represent $\sqrt{-1}$.

$$z = a + bi \quad \text{(or equivalently, } z = a + ib\text{)},$$

where $i = \sqrt{-1}$, and a and b are real numbers. We refer to a as the **real part** of z, written $\operatorname{Re}(z)$, and to b as the **imaginary part** of z, written $\operatorname{Im}(z)$. A complex number of the form $a + 0i$ is, in effect, just the real number a; so the real numbers are seen as a subset of the complex numbers.

$\operatorname{Im}(z)$ is the *real* number b; it is *not* equal to bi.

The set of all complex numbers is denoted by $\mathbb{C}$. Within $\mathbb{C}$, we can solve any quadratic equation. For example, the equation $x^2 - 2x + 2 = 0$ has the solutions

$$x = \frac{2 \pm \sqrt{2^2 - 4 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4} \times \sqrt{-1}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i,$$

and the equation $x^2 = -1$ has the solutions $x = \pm i$.

An ***n*th-order polynomial** with real coefficients is a function of the form

An nth-order polynomial is sometimes referred to as a **polynomial of degree n**.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$ and each coefficient a_k $(k = 0, 1, \ldots, n)$ is a constant in $\mathbb{R}$.

If x is allowed to be complex, any such polynomial can be written as a product of the form

In fact, this result also holds if the coefficients a_k are complex.

$$p(x) = a_n (x - c_1)(x - c_2) \ldots (x - c_n),$$

where each c_k $(k = 1, 2, \ldots, n)$ is a complex number. Remember that real numbers are included in the complex numbers, so this does not prevent some (or all) of the c_k being real. The equation $p(x) = 0$ then has n solutions: $x = c_1$, $x = c_2$, …, $x = c_n$. These are called the **roots** of the polynomial. If a factor $x - c$ occurs more than once, then the root c is a **repeated root**. For example, the polynomial $x^2 - 2cx + c^2 = (x - c)^2$ has the repeated root c.

Repeated roots are sometimes referred to as **equal roots** or **coincident roots**.

Why are complex numbers important?

Complex numbers are an abstract construct, seemingly far removed from what you can measure with laboratory apparatus. Their mystical air of unreality is scarcely helped by calling $\sqrt{-1}$ an *imaginary* number! It is true that you cannot use complex numbers to count sheep, but complex numbers describe other aspects of reality and are used in numerous descriptions of physical phenomena.

One reason why complex numbers are important is that their exponentials (discussed below) are closely linked to the oscillating functions that describe vibrations and waves. But beyond this, it seems that complex numbers are woven deeply into fundamental scientific laws. In fact, the laws of quantum mechanics, our most fundamental theory of Nature, are expressed in terms of complex numbers.

4.1 The arithmetic of complex numbers

We can perform arithmetic with complex numbers, and this follows all the familiar rules for real numbers, such as

$$u(v+w) = uv + uw \quad \text{and} \quad u \times v = v \times u.$$

To add, subtract or multiply complex numbers, just manipulate brackets in the usual way, and remember that $i^2 = -1$. For example,

$$(2+3i) + (4-7i) = 2 + 4 + 3i - 7i = 6 - 4i$$

and

$$\begin{aligned}(2+3i) \times (4-7i) &= 2 \times (4-7i) + 3i \times (4-7i)\\ &= 8 - 14i + 12i - 21i^2\\ &= 8 + 21 - 2i\\ &= 29 - 2i.\end{aligned}$$

Division of complex numbers is a little more complicated. It is best described in terms of the **complex conjugate**, which is defined as follows: if $z = a + bi$ is a complex number, then the complex conjugate of z is

$$\overline{z} = a - bi.$$

You will see the notation z^* for the complex conjugate of z in many textbooks.

Then, to divide one complex number by another, as in u/v, we multiply top and bottom by the complex conjugate $\overline{v}$ of the denominator. For example, to simplify

$$\frac{2+3i}{4-7i},$$

we multiply top and bottom by $4 + 7i$, the complex conjugate of the denominator, to obtain

$$\begin{aligned}\frac{2+3i}{4-7i} &= \frac{(2+3i)\times(4+7i)}{(4-7i)\times(4+7i)}\\ &= \frac{8+14i+12i-21}{16+28i-28i+49}\\ &= \frac{-13+26i}{65}\\ &= -\tfrac{1}{5}+\tfrac{2}{5}i.\end{aligned}$$

This process always reduces the denominator to a real number because the product of a complex number $a+bi$ and its complex conjugate $a-bi$ is always real:

Note that a^2+b^2 is always positive, unless $a=b=0$.

$$(a+bi)\times(a-bi)=a^2+b^2.$$

Given any complex number $z=a+bi$, its **modulus** $|z|$ is defined to be $\sqrt{a^2+b^2}$, so we have

$$|z|^2=a^2+b^2=z\overline{z}.$$

The process of dividing one complex number, $u=a+bi$, by another, $v=c+di$, is then summarised by

$$\frac{u}{v}=\frac{u\overline{v}}{v\overline{v}}=\frac{u\overline{v}}{|v|^2}=\frac{(a+bi)\times(c-di)}{c^2+d^2}. \tag{48}$$

Exercise 17

Let $v=3-4i$ and $w=2-i$. Evaluate each of the following.

(a) $\overline{v}$ (b) $|v|$ (c) $v-w$ (d) vw

(e) w/v (f) $1/w$ (g) w^2 (h) $2w-3v$

Exercise 18

Solve (for x in $\mathbb{C}$) the quadratic equation $2x^2+2x+1=0$.

Notice that the solutions obtained in Exercise 18 are complex conjugates of one another. This is not accidental: any quadratic function with real coefficients has roots that are a pair of complex conjugates (of the form $a\pm bi$). This follows directly from equation (9) for the solution of a quadratic equation.

4.2 Polar form of complex numbers

Diagnostic test

Try Exercises 21(a), 21(c) and 21(f). If your answers agree with the solutions, you may proceed quickly to Subsection 4.3.

Before discussing the polar form of a complex number, here is a brief review of polar coordinates.

Polar coordinates

Polar coordinates provide an alternative way of representing points in the plane. Figure 21 shows a point A with Cartesian coordinates (x, y) and polar coordinates (r, θ). The quantity r is the distance from the origin to A, so $r \geq 0$. The angle θ is measured anticlockwise from the positive x-axis. (Negative angles correspond to measuring clockwise from the positive x-axis.)

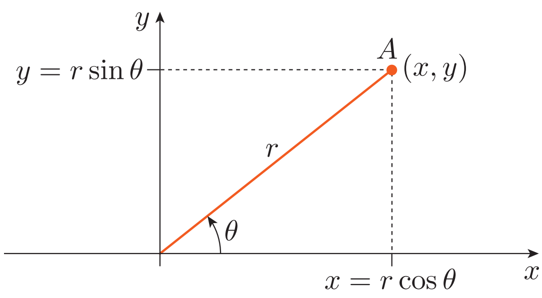


Figure 21 Cartesian (x, y) and polar (r, θ) coordinates of a point A

It is convenient to allow θ to take any real value, but this has the consequence that the polar representation of a point is not unique. For example, (r, θ) and $(r, \theta + 2\pi)$ provide polar coordinates of the same point. We can see from Figure 21 that if a point has polar coordinates (r, θ) and Cartesian coordinates (x, y), then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{49}$$

These equations allow us to translate from polar to Cartesian coordinates. To translate from Cartesian to polar coordinates, we can use (see Figure 21)

$$r = \sqrt{x^2 + y^2}, \quad \cos\theta = x/r, \quad \sin\theta = y/r \quad (r \neq 0). \tag{50}$$

If $r = 0$, then we can choose any value for θ.

Equations (50) do not have a unique solution for θ in $\mathbb{R}$, but they do have a unique solution in the range $-\pi < \theta \leq \pi$.

Exercise 19

Locate each of the following points (x, y) on a diagram like Figure 21.

$$(-2, 0), \quad (1, 1), \quad (-1, -1), \quad (4, 0), \quad (0, 4), \quad (-\sqrt{3}, 1).$$

Hence find the polar coordinates (r, θ) of each point, for θ in the range $-\pi < \theta \leq \pi$.

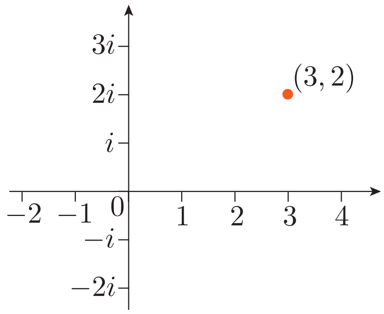


Figure 22 Argand diagram showing the point $3 + 2i$

Polar form of a complex number

A complex number $x + yi$ can be represented geometrically by treating its real and imaginary parts as Cartesian coordinates (x, y) in a plane. This produces a diagram known as an **Argand diagram**. For example, Figure 22 shows on an Argand diagram the point $3 + 2i$, with real part 3 and imaginary part 2.

By combining polar coordinates with the Argand diagram, we obtain the *polar form* of a complex number. For a complex number $z = x + yi$, we take the Cartesian coordinates (x, y) and convert them to the corresponding polar coordinates (r, θ). Then, using the relation between polar and Cartesian coordinates, we have

$$z = x + yi = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta). \qquad (51)$$

This is the **polar form** of z. Here, $r = \sqrt{x^2 + y^2} = |z|$ is the **modulus** of z, and θ is an **argument** of z. As noted above, θ is not unique, but there *is* a unique value of θ in the range $-\pi < \theta \leq \pi$. This is called the **principal value** of the argument, and is written as $\mathrm{Arg}(z)$. When there is no possibility of confusion, we often write (r, θ) as shorthand for the polar form $r(\cos\theta + i\sin\theta)$.

Exercise 20

If a complex number z has polar form $\left(2, -\frac{\pi}{4}\right)$, what is its Cartesian form?

Exercise 21

Express each of the following complex numbers in polar form, choosing the principal value of the argument.

(a) -2 (b) $1 + i$ (c) $-1 - i$ (d) 4 (e) $4i$ (f) $-\sqrt{3} + i$

4.3 Euler's formula

Diagnostic test

Try Exercises 23 and 24. If your answers agree with the solutions, you may proceed quickly to Section 5.

The definition of the exponential function, equation (18), remains valid even when x is a complex number. There is a remarkable result that connects the exponential function and trigonometric functions, known as Euler's formula. If x is a real number, **Euler's formula** states that

$$\exp(ix) = \cos x + i\sin x. \qquad (52)$$

To see why this is true, replace x in equation (18) by ix. This gives the Taylor series for $\exp(ix)$:

$$\exp(ix) = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - \cdots . \qquad (53)$$

Note that the definition $i^2 = -1$ gives $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, etc.

Collecting together the real and imaginary parts, we see that they correspond to the Taylor expansions of $\cos x$ and $\sin x$ (equations (44) and (45)) introduced at the end of Subsection 3.3:

$$\exp(ix) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

So

$$\exp(ix) = \cos x + i\sin x, \quad \text{or equivalently,} \quad e^{ix} = \cos x + i\sin x.$$

'Our jewel'

Euler's formula is a very interesting result because it connects exponential functions, which are easy to deal with mathematically, with trigonometric functions, which occur in descriptions of waves, oscillations and vibrations. Richard Feynman (Figure 23), who was a noted educator as well a Nobel prizewinner for his work on quantum electrodynamics, referred to Euler's formula as 'our jewel'.

One of the most common uses of Euler's formula is in the description of wave motion (Figure 24). A wave, such as an undulation of the height h of the surface of some water at position x and time t, might be described by a function such as

$$h(x,t) = A\cos(kx - \omega t),$$

where k, ω and A are real constants. In advanced calculations involving wave motion (including the ones that made Richard Feynman famous), a wave is often represented by a complex function of the form

$$H(x,t) = A\exp[i(kx - \omega t)],$$

whose real part is given by the equation for $h(x,t)$ above. It is often convenient to use this exponential form, and then take the real part at the end of the calculation.

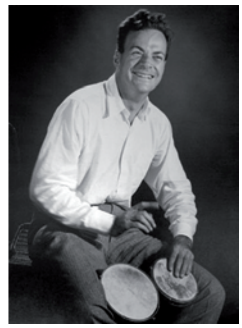


Figure 23 Richard Feynman (1918–1988), Nobel laureate and bongo player

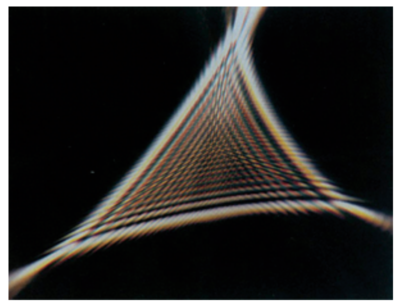


Figure 24 Euler's formula can help us to understand subtle phenomena involving wave motion, such as this pattern caused by the partial focusing of light waves

Exponential form of complex numbers

Combining equation (51) for the polar form of a complex number with Euler's formula, equation (52), it follows that any complex number can be written as

$$z = x + iy = r\exp(i\theta),$$

where r and θ have the same values as in the polar form. This alternative way of expressing a complex number, the **exponential form**, is particularly useful when we need to multiply and divide complex numbers.

In this form, r is the modulus of z, and θ is the argument of z. As with the polar form, the value of θ is not unique, but there is a unique choice of θ in the range $-\pi < \theta \leq \pi$.

Exercise 22

Express the following complex numbers as complex exponentials with arguments in the range $-\pi < \theta \leq \pi$.

(a) $z = 3e^{-i7\pi/2}$ (b) $z = e^{i71\pi}$

Exercise 23

A complex number has polar form $z = (r, \theta)$. Use the exponential form of z to find $\operatorname{Re}(ze^{i\omega t})$, i.e. the real part of $ze^{i\omega t}$, where ω and t are real.

Multiplication, division and powers revisited

Multiplication and division of complex numbers is far simpler in exponential and polar forms than in Cartesian form. If $z_1 = x_1 + iy_1 = r_1e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = (r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)\,e^{i(\theta_1+\theta_2)}. \tag{54}$$

Note that although $\theta_1 + \theta_2$ is *an* argument of the product, it may not be the principal value of the argument.

So to multiply two numbers in exponential form, we just multiply their moduli and add their arguments. In polar form this rule is written as

$$\begin{aligned}&r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)\\&= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)),\end{aligned}$$

or, in shorthand notation,

$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1r_2, \theta_1 + \theta_2).$$

Similarly, for division of complex numbers,

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}. \tag{55}$$

So to divide two numbers in exponential form, we just divide their moduli and subtract their arguments. In shorthand polar form this means that

$$(r_1, \theta_1) \div (r_2, \theta_2) = (r_1/r_2, \theta_1 - \theta_2).$$

Further, if we multiply the complex number $z = re^{i\theta}$ by itself repeatedly, we obtain a formula for an integer power of a complex number:

$$z^n = (re^{i\theta})^n = r^ne^{in\theta} \tag{56}$$

or

$$(r, \theta)^n = (r^n, n\theta).$$

For a complex number of unit modulus (i.e. one with $r = 1$), we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{57}$$

This result is known as **de Moivre's theorem** (after Abraham de Moivre, 1667–1754).

To avoid problems, n must be an integer in de Moivre's theorem.

Exercise 24

Express $1 - i$ in exponential form and hence simplify $(1 - i)^{20}$.

Euler's formula also leads to useful expressions for the trigonometric functions in terms of complex exponentials. A complex number of unit modulus can be written as

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and its complex conjugate is

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

The complex conjugate is obtained by changing the sign of i wherever it appears.

Adding and subtracting these two equations then gives the results

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \tag{58}$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \tag{59}$$

These formulas find many applications. As an example, let us derive a trigonometric identity relating $\cos^2\theta$ to $\cos 2\theta$. Using equation (54), we have

$$\begin{aligned}\cos^2\theta &= \tfrac{1}{4}\left(e^{i\theta} + e^{-i\theta}\right)^2\\ &= \tfrac{1}{4}\left(e^{i\theta}e^{i\theta} + 2e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{-i\theta}\right)\\ &= \tfrac{1}{4}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right)\\ &= \tfrac{1}{4}\left(e^{2i\theta} + e^{-2i\theta}\right) + \tfrac{1}{2}\\ &= \tfrac{1}{2}[\cos(2\theta) + 1].\end{aligned}$$

This is a rearranged version of equation (42).

5 Differentiation

The concepts and techniques of calculus are central to many of the mathematical methods discussed in this module. This section considers differentiation.

Differentiation is fundamental to mechanics: if an object moving in a straight line has position $x(t)$ at time t, then its velocity at time t is given by the derivative

$$v(t) = \frac{dx}{dt},$$

and its acceleration at time t is given by

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

5.1 Derivative as a rate of change

You may use the Handbook to look up standard derivatives.

Diagnostic test

Try Exercises 27(d) and 27(e). If your answers agree with the solutions, you may proceed quickly to Subsection 5.2.

Differentiation gives the rate of change of one variable with respect to another. For example, consider the function (introduced in Subsection 1.1)

$$V(t) = 2 \times 10^6 - 15t, \tag{60}$$

where $V(t)$ is the volume of water in a reservoir (in cubic metres) at time t (in minutes, from midday on 1 June). In this case, the rate of change of V with respect to t is -15 (in cubic metres per minute). This is negative because the volume is decreasing, and it is constant because the volume of water is falling by the same amount each minute. For linear functions, such as this, the rate of change is always constant. This corresponds to the fact that a linear function has a straight-line graph whose gradient (or slope) is the same everywhere.

Now consider a non-linear function, such as

$$y(t) = -4.9t^2 + 10t + 2 \tag{61}$$

(introduced in Subsection 1.3), which gives the height (y metres at time t seconds) of a ball thrown vertically upwards with an initial speed of 10 metres per second, from an initial height of 2 metres. In this case, the rate of change of height with respect to time is the vertical velocity v of the ball (which is positive when the ball is moving upwards, and negative when it is moving downwards). Differentiation allows us to show that the velocity of the ball (in metres per second) is

$$v(t) = -9.8t + 10. \tag{62}$$

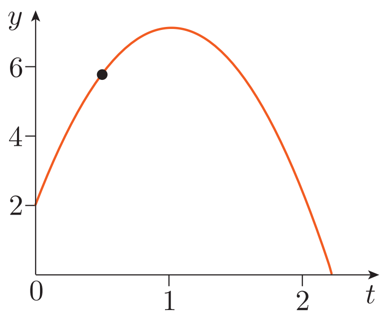


Figure 25 A graph of height y against time t for a ball thrown upwards

This is a function of time: the velocity of the ball is positive to begin with, and then becomes negative as the ball falls back to the ground. This corresponds to the fact that the quadratic function in equation (61) has a parabolic graph (Figure 25), with a gradient that varies from point to point. The process that takes us from the function $y(t)$ in equation (61) to the function $v(t)$ in equation (62) is called **differentiation**. The function $v(t)$ is called the **derivative** or **derived function** of $y(t)$. This function is denoted by dy/dt or $y'(t)$, so in the present case

$$\frac{dy}{dt} = y'(t) = -9.8t + 10.$$

At each value of t, this derivative gives the gradient of the graph of y against t.

More generally, given a function $f(x)$, the gradient of the graph of $f(x)$ against x at a particular point $x = x_0$ is equal to the derivative $f'(x)$ of the function f at that point. This leads to a definition of the derivative, based on Figure 26.

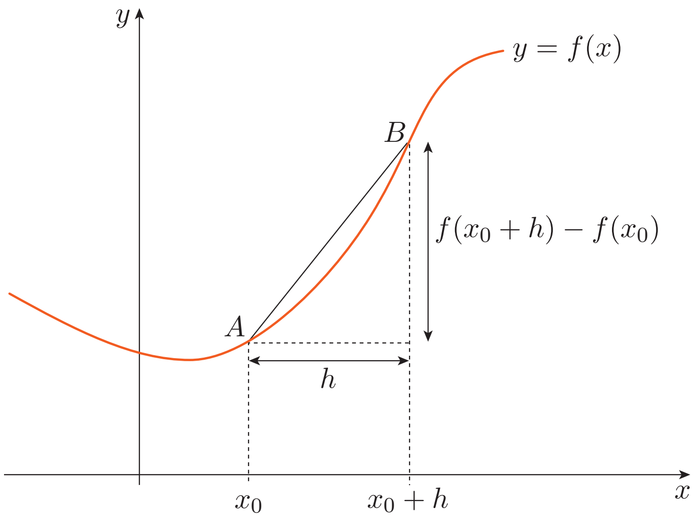


Figure 26 The gradient of f at x_0, defined as the limiting value of the gradient of the chord AB

The **gradient** of the graph $y = f(x)$ at $x = x_0$ is defined as the limiting value of the gradient of the chord AB in Figure 26, as B approaches A. The gradient of this chord is $(f(x_0 + h) - f(x_0))/h$, and the process of allowing B to approach A corresponds to h tending to 0, often written as $h \to 0$. Hence the derivative of f at x_0 may be formally defined as follows.

This definition assumes that the limit exists and is the same whether h approaches 0 through positive or negative values. When these conditions are not met, the function cannot be differentiated at x_0.

Definition

The **derivative of a function** $f(x)$ at $x = x_0$ is

$$f'(x_0) = \lim_{h\to 0}\left(\frac{f(x_0 + h) - f(x_0)}{h}\right).$$

Working from this definition, it is possible to obtain formulas for the derivatives of standard functions. You need not be concerned with the sometimes lengthy details. For our purposes, it is enough to know that all the basic derivatives that we need are tabulated in the module Handbook, and can be used as required. As usual, it helps to remember the most frequently used results, such as the derivatives of x^a, $\sin x$, $\exp(x)$ and $\ln x$, but this generally comes with practice rather than rote memorisation. Note that the *basic* derivatives are tabulated. The Handbook does not give the derivative of every function that you might encounter. Generally, you need to combine two elements:

- derivatives of standard functions
- rules for differentiating combinations of functions of various types, in sums, products, quotients and compositions.

The skill in differentiation lies mostly in applying the rules. The simplest rule concerns constant multiples and sums. In general, the derivative of a

combination $a\,f(x) + b\,g(x)$, where a and b are constants, is given by

$$\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\,\frac{df}{dx} + b\,\frac{dg}{dx}. \tag{63}$$

To illustrate this, let us return to equation (61):

$$y(t) = -4.9t^2 + 10t + 2.$$

Applying patterns for derivatives given in the Handbook, we see that

$$\frac{d}{dt}(t^2) = 2t, \quad \frac{d}{dt}(t) = 1 \quad \text{and} \quad \frac{d}{dt}(\text{constant}) = 0.$$

Hence, applying the rule in equation (63), we get

$$\begin{aligned}\frac{dy}{dt} &= -4.9\,\frac{d}{dt}(t^2) + 10\,\frac{d}{dt}(t) + \frac{d}{dt}(2)\\ &= -4.9(2t) + 10(1) + 0\\ &= -9.8t + 10.\end{aligned}$$

A similar process applied to the function in equation (60) gives $dV/dt = -15$, a constant negative value corresponding to the fact that V decreases at a constant rate.

Notation for derivatives

There are various notations for derivatives, some of which we have used above. We will use whichever is convenient in a particular context.

In text, $\dfrac{dy}{dt}$ may be written as dy/dt, to save space.

Notation expressed purely in terms of variables, such as dy/dt, is referred to as *Leibniz notation* (after its inventor, Gottfried Wilhelm Leibniz (1646–1716)). This notation is extended to write, for example,

$$\frac{d}{dt}(3t + 5\sin 2t) \quad \text{or} \quad \frac{d}{dx}(ax + bx^2).$$

It is implicit that, in general, dy/dt is a function of t, but this fact may sometimes be emphasised by writing $\dfrac{dy}{dt}(t)$. In Leibniz notation, the value of a derivative at a particular point, such as $t = 3$, is written as $\left.\dfrac{dy}{dt}\right|_{t=3}$.

The alternative to Leibniz notation is *function notation*, where differentiation is indicated by adding a prime ($'$) to the function name. So the derivative of $f(x)$ is $f'(x)$. This clearly shows that the derivative is a function of x, and the value of the derivative at $x = 3$ is written as $f'(3)$. To give another example, the derivative of $h(t)$ is $h'(t)$, and the value of this derivative at $t = 4$ is $h'(4)$. We must take care to use this notation correctly. The symbol y', for example, means the derivative of the function y, but could lead to confusion if y is being used for both $y(t)$ and $y(x)$.

Because the derivative of a function is itself a function, it can be differentiated again. For example, if $y(x) = x^3 + 5x$, then

$$\frac{dy}{dx} = y'(x) = 3x^2 + 5. \tag{64}$$

This derivative is itself a function of x, and can be differentiated again to obtain the **second derivative** of $y(x)$. In Leibniz notation, we write the second derivative as d^2y/dx^2. In function notation, it is written with two primes, as $y''(x)$. So the derivative of equation (64) is

The derivative dy/dx is sometimes referred to as the **first derivative**.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''(x) = 6x.$$

The value of this derivative at $x = 4$ is then denoted by

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = 24 \quad \text{or} \quad y''(4) = 24.$$

Differentiating again gives the **third derivative**, written $\dfrac{d^3y}{dx^3}$ or $y'''(x)$, which may also be written $y^{(3)}(x)$. For equation (64), $y'''(x) = 6$. The process can be continued, and a general ***n*th derivative** is written as $\dfrac{d^ny}{dx^n}$ or $y^{(n)}(x)$, where n is referred to as the **order** of the derivative.

There is one final piece of notation to mention. Time (habitually denoted by t) is so often the independent variable that there is a separate notational convention for differentiation with respect to it. A dot is placed over the dependent variable to indicate a first derivative with respect to t, and two dots to indicate a second derivative. So if $x(t)$ is the position of an object as a function of time t, then $\dot{x}$ means the same as dx/dt or $x'(t)$, while $\ddot{x}$ means the same as d^2x/dt^2 or $x''(t)$. As noted at the beginning of this section, these derivatives represent the velocity and acceleration of a particle in straight-line motion.

This notation is attributed to Isaac Newton (1642–1727), so is sometimes referred to as *Newtonian notation*.

The following exercises offer practice in differentiating standard functions, and constant multiples and sums of these. Refer to the Handbook for standard derivatives if necessary.

It will be helpful later on, especially in integration, if you can remember the derivatives of polynomials, exponentials, natural logarithms and trigonometric functions.

Exercise 25

Suppose that an object is moving in a straight line so that its position x (measured from a chosen origin) is related to time t by the equation

$$x = 5 + 7\cos(3t).$$

Find expressions in terms of t for the velocity $\dot{x}$ and acceleration $\ddot{x}$ of the object.

Exercise 26

The weekly wage bill of a company, t years in the future, is projected to be B pounds sterling, where

$$B = 10^5 \exp(0.04t).$$

Find an expression for the rate at which the wage bill will be rising in t years' time. What will this rate of rise be as a percentage of the wage bill at that time?

Exercise 27

Calculate the following derivatives.

(a) $\dfrac{dy}{dx}$, where $y = 1 - 9\exp(-5x)$.

(b) $F'(2)$, where $F(x) = 3x^4 - 4x + 1$.

(c) $\dfrac{d^2y}{dt^2}$, where $y = \ln t$ $(t > 0)$.

(d) $g''(0)$, where $g(t) = a\cos(3t) + b\sin(3t)$ (and a and b are constants).

(e) $F'\left(\frac{\pi}{6}\right)$, where $F(x) = 3\sec(2x) - 4\cos(-3x)$.

We sometimes need to differentiate a **complex-valued** function of the form

$$f(t) = g(t) + i\,h(t),$$

where g and h are real functions. Differentiation of such a function follows the usual rule for constant multiples and sums. So

$$f'(t) = g'(t) + i\,h'(t).$$

For example, if $f(t) = \cos(3t) + i\sin(3t)$, then

$$f'(t) = -3\sin(3t) + 3i\cos(3t).$$

Exercise 28

Find the second derivative d^2f/dt^2 of the function $f(t) = \cos(2t) + i\sin(2t)$. Do this in two ways: first by differentiating the real and imaginary parts, and then by using Euler's formula to express this function as a complex exponential, which you can differentiate using equation (19). Check that you obtain the same result in both cases.

5.2 Differentiating combinations of functions

Diagnostic test

Try Exercises 31 and 32(b). If your answers agree with the solutions, you may proceed quickly to Subsection 5.3.

The rule for differentiating constant multiples and sums is natural and easy to apply. Rules for products, quotients and compositions of functions are also very important. They are given below and in the Handbook, but you will be much better off if you remember them through practice of use. We begin with the rules for products and quotients.

The product rule

The derivative of the product of two functions is given by

$$(fg)' = f'g + fg', \tag{65}$$

or, in Leibniz notation,

$$\frac{d}{dx}(uv) = \frac{du}{dx}\,v + u\,\frac{dv}{dx}. \tag{66}$$

The quotient rule

The derivative of the quotient of two functions is given by

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \tag{67}$$

or, in Leibniz notation,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\dfrac{du}{dx}\,v - u\,\dfrac{dv}{dx}}{v^2}. \tag{68}$$

Example 3

(a) Find $h'(x)$, where $h(x) = x^3\cos(2x)$.

(b) Find $w'(x)$, where $w(x) = x^2/(3x^2+1)$.

Solution

(a) $h(x)$ is a product $f(x)\,g(x)$, with $f(x) = x^3$ and $g(x) = \cos(2x)$.

We have

$$f'(x) = 3x^2 \quad \text{and} \quad g'(x) = -2\sin(2x).$$

So the product rule gives

$$h'(x) = 3x^2\cos(2x) - 2x^3\sin(2x).$$

(b) $w(x)$ is a quotient $u(x)/v(x)$, with $u(x) = x^2$ and $v(x) = 3x^2+1$.

We have

$$u'(x) = 2x \quad \text{and} \quad v'(x) = 6x.$$

So the quotient rule gives

$$w'(x) = \frac{2x(3x^2+1) - x^2 \times 6x}{(3x^2+1)^2} = \frac{2x}{(3x^2+1)^2}.$$

Exercise 29

(a) Find $\dfrac{dy}{dx}$, where $y(x) = \dfrac{\ln x}{x^2+1}$.

(b) Find $f'(t)$, where $f(t) = t^5 \ln(3t)$ and $t > 0$.

(c) Find $g'(0)$ (in terms of the constants A, B and C), where $g(t) = (At + B)\sin(Ct)$.

(d) If the position of an object at time t is given by $e^{-3t}\sin(4t)$, find its velocity and acceleration as functions of time.

The rule for composite functions is a little more complicated to use.

The composite rule

If h is the composition of two functions g and u, so that $h(x) = g(u(x))$, then

$$h'(x) = g'(u(x))\,u'(x). \tag{69}$$

Expressed in Leibniz notation, this rule looks rather different: if $u = u(x)$ is a function of x, and $h = h(u)$, then

$$\frac{dh}{dx} = \frac{dh}{du}\,\frac{du}{dx}. \tag{70}$$

In this form, the composite rule is often called the **chain rule**.

The essential point is that the derivative of a composite function is found by differentiating the outer function and then multiplying by the derivative of the inner function. For example, consider differentiating $h(x) = \sin(x^3 + 2x)$. First, you should recognise that $h(x)$ is the composition of two functions: $h(x) = g(u(x))$. The inner function is $u(x) = x^3 + 2x$, and the outer function is $g(u) = \sin(u)$. To apply the composite rule, we first find the derivatives of these: $u'(x) = 3x^2 + 2$ and $g'(u) = \cos(u)$. Multiplying these then gives

$$h'(x) = g'(u(x)) \times u'(x) = \cos(x^3 + 2x) \times (3x^2 + 2).$$

Alternatively, we can use the chain rule by writing $u = x^3 + 2x$ so that $h(u) = \sin(u)$. Then

$$\frac{dh}{dx} = \frac{dh}{du}\,\frac{du}{dx} = \cos(u) \times (3x^2 + 2) = \cos(3x^2 + 2) \times (3x^2 + 2).$$

Example 4

Find df/dx, where $f(x) = \sin^3 x = (\sin x)^3$.

Solution

If we let $u = \sin x$, then we have $f(x) = [u(x)]^3$.

The recognition of $\sin^3 x$ as a composite function, and of how to break it down into two parts, each consisting of a standard function, is the key to differentiating it.

The chain rule then gives

$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx} = 3u^2 \cos x = 3\sin^2 x \cos x,$$

where we have replaced the variable u by $\sin x$ in the last step.

Exercise 30

Use the composite rule (or chain rule) to differentiate each of the following.

(a) $y(t) = \exp(t^2)$ (b) $f(x) = (3x^3 + 4)^6$ (c) $z(v) = \tan(3v + 4)$

(d) $g(z) = \sqrt{4 - z^2}$ (e) $f(x) = 1/\left(\sqrt{1 + 2x^2}\,\right)^3$

Exercise 31

Differentiate the following functions.

These differentiations involve more than one rule.

(a) $y = \sec\left(\dfrac{x}{x^2 + 1}\right)$ (b) $z = t^2 \exp(t^3 + 1)$

Implicit differentiation

Suppose that we want to find the gradient at the point $(x, y) = (2, 1)$ of the tangent to the ellipse with equation

$$x^2 + 4y^2 = 8. \tag{71}$$

We want dy/dx at $x = 2$ and $y = 1$. We could start by expressing y as a function of x, but a more convenient approach is to differentiate the equation as it stands. To differentiate y^2 with respect to x, we use the composite rule, and obtain

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy}\,\frac{dy}{dx} = 2y\,\frac{dy}{dx}.$$

So, differentiating both sides of equation (71) with respect to x, we obtain

$$2x + 4 \times 2y\,\frac{dy}{dx} = 0.$$

When $x = 2$ and $y = 1$, this gives $4 + 8\,dy/dx = 0$, so $dy/dx = -\frac{1}{2}$. Therefore the tangent to this ellipse at $(2, 1)$ has gradient $-\frac{1}{2}$.

Differentiation with respect to x of an expression such as $x^2 + 4y^2$, where y is a function of x, is known as **implicit differentiation**.

Exercise 32

(a) Use the product and composite rules to find the following in terms of x, y and dy/dx.

(i) $\dfrac{d}{dx}(x^2y)$ (ii) $\dfrac{d}{dx}(y^3)$ (iii) $\dfrac{d}{dx}(x+\sin(xy))$

(b) Find the gradient at the point $(-1, 1)$ of the tangent to the curve

$$x^3 + x^2y + y^3 = 1.$$

5.3 Investigating functions

Diagnostic test

Try Exercise 33. If your answer agrees with the solution, you may proceed quickly to Section 6.

Faced with an expression made up of some combination of standard functions, how might you investigate its behaviour? As an example, consider the function

$$f = \frac{v}{4 + 1.5v + 0.008v^2},$$

where we would like to see how f varies as v varies.

A sketch graph of f against v helps with this, and a computer package or graphics calculator will provide such a graph. However, it is not always obvious for what range of values to plot the graph, so it is helpful to be able to deduce some information about the general behaviour of a function 'by hand', without recourse to a machine. Such information can also be used to cross-check results obtained from a machine, and to flesh out the picture more fully. This example will be continued in Exercise 34, but first consider some general remarks about sketching graphs.

Example 5 below illustrates ways of answering these questions.

Questions that you might consider before sketching the graph of a function $f(x)$ include the following:

1. Are there any points where $f(x)$ is not defined?
2. Where does the graph of $f(x)$ cross the axes?
3. How does $f(x)$ behave for large and small values of x (or at the endpoints of its domain if this is finite)?
4. Are there any stationary points of $f(x)$? If so, are there any local maximum or minimum values?
5. Where does the graph of $f(x)$ have a positive gradient, and where does it have a negative gradient?

The last two questions can be answered using differentiation.

Definition

A **stationary point** of a function $f(x)$ is a value of x where $f'(x) = 0$.

Local maxima and local minima occur at stationary points, although a stationary point need not necessarily be either. Figure 27 illustrates stationary points of various kinds.

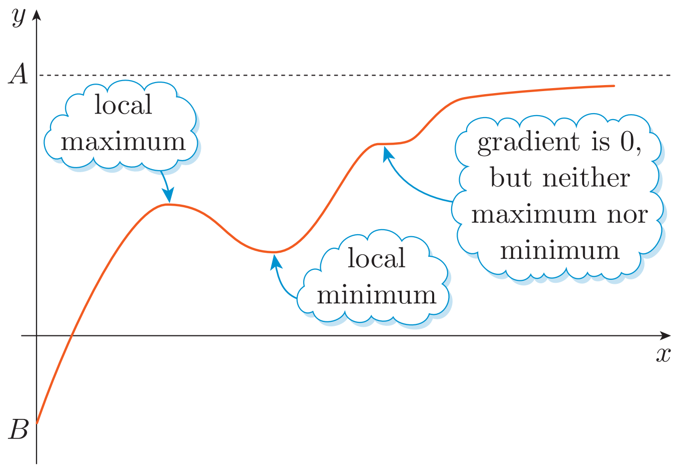


Figure 27 Stationary points of a hypothetical function $f(x)$

There is a **local maximum** at x_0 if $f(x_0) > f(x)$ for all points x in the immediate vicinity of x_0, and there is a **local minimum** at x_0 if $f(x_0) < f(x)$ for all points x in the immediate vicinity of x_0. Figure 27 also shows a stationary point that is neither a local maximum nor a local minimum: such a point is called a **point of inflection**.

To classify a given stationary point of $f(x)$, we can use a test based on the second derivative, $f''(x)$.

Classifying stationary points using second derivatives

A stationary point x_0 of a function $f(x)$ is:

- a local maximum if $f''(x_0) < 0$
- a local minimum if $f''(x_0) > 0$
- a point of inflection if $f''(x_0) = 0$ *and* $f''(x)$ changes sign as x increases through x_0.

Note that the condition $f''(x_0) = 0$ is insufficient *by itself* to determine the nature of the stationary point.

An alternative strategy is sometimes preferred; it is useful in cases where evaluation of the second derivative is messy.

Classifying stationary points using first derivatives

A stationary point x_0 of a function $f(x)$ is:

- a local maximum if, for all x in the immediate vicinity of x_0, we have $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$
- a local minimum if, for all x in the immediate vicinity of x_0, we have $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$.

We often wish to find the overall maximum or overall minimum of some function, usually referred to as the **global maximum** or **global minimum**, respectively. The global minimum or global maximum of a function may well occur at a stationary point; but caution is needed, for it need not necessarily do so. For example, if the function $f(x)$ in Figure 27 is defined in the domain $0 \leq x < \infty$, then its global minimum occurs at the endpoint $x = 0$, which is not a stationary point. In fact, a function need not have a global maximum or global minimum. For example, the function in Figure 27 exceeds the local maximum value when x is large, but it never reaches a global maximum. Notice that the value of $f(x)$ does not grow without limit. Instead, it gets arbitrarily close to the value A as x increases, but it always remains smaller than A, and never actually reaches this limiting value. The line $y = A$ is called an **asymptote** of the graph of $f(x)$. Such behaviour is sometimes indicated by writing $f \to A$ as $x \to \infty$, which is read as 'f tends to the value A as x tends to infinity'.

Exercise 33

(a) Find any stationary points of the function

$$y(x) = 5 - 2(x+1)e^{-x/2} \quad (x \geq 0).$$

(b) Classify these as local minima or local maxima or neither, and evaluate $y(x)$ at these points.

Example 5

Suppose that

$$(x^2 - 3)y = x - 2.$$

Sketch a graph of y against x.

Solution

We address some of the questions listed at the beginning of this subsection.

(1) *Points where y is not defined.* We have

$$y = \frac{x-2}{x^2-3}.$$

This is not defined when $x^2 - 3 = 0$, i.e. at $x = \pm\sqrt{3}$.

(2) *Points where $y(x)$ crosses the axes.* We can see that $y = 0$ if (and only if) $x = 2$, so the graph crosses the x-axis at this one point.

(3) *Behaviour for large values of x.* If x is large (positive or negative), then y will be close to zero.

(4) *Stationary points and their classification.* To look for stationary points, we use the quotient rule to calculate

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1)(x^2-3)-(x-2)(2x)}{(x^2-3)^2} \\ &= \frac{-x^2+4x-3}{(x^2-3)^2} = -\frac{(x-1)(x-3)}{(x^2-3)^2}.\end{aligned}$$

This is zero if $(x-1)(x-3) = 0$. So the function has stationary points at $x = 1$ and $x = 3$.

In this case, the second derivative is a bit messy to calculate, and it is easier to look at the sign of the first derivative near $x = 1$ and $x = 3$ to check whether these stationary points are local maxima or minima. If x is just less than 1, then dy/dx is negative, while if x is just greater than 1, then dy/dx is positive. Hence $x = 1$ is a local minimum. For x just below 3, dy/dx is positive, while for x just above 3, it is negative, so $x = 3$ is a local maximum. The values of the function at these points are: $y = 1/2$ at $x = 1$, and $y = 1/6$ at $x = 3$.

Try $x = 0.9$ and $x = 1.1$.

Try $x = 2.9$ and $x = 3.1$.

(5) *Regions of positive and negative gradient.* Just below the local minimum, and just above the local maximum, the function has a negative gradient. Just above the local minimum, and just below the local maximum, it has a positive gradient. Because there are no other local minima or maxima, we can make further deductions: for example, the gradient remains positive everywhere between the local minimum at $x = 1$ and the point $x = \sqrt{3}$, where the function is not defined.

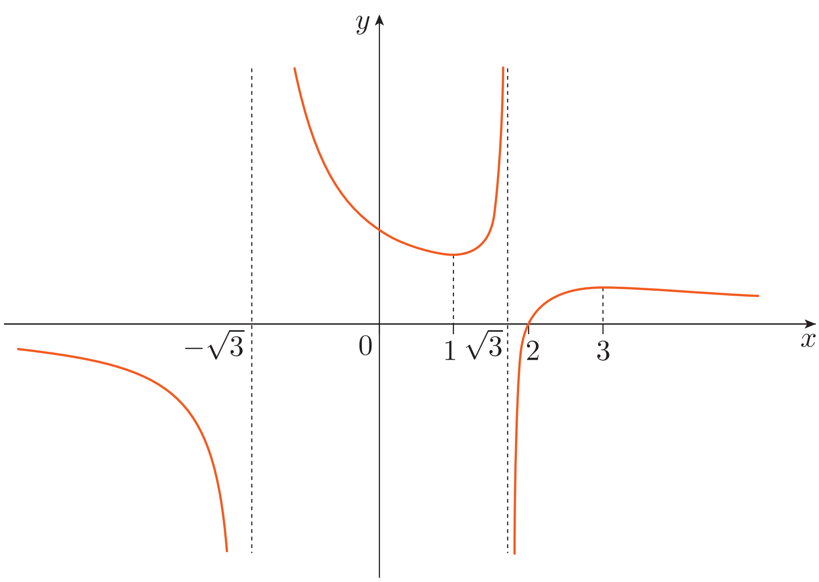


Figure 28 A graph of $y = (x-2)/(x^2-3)$

Incorporating all this information (plus other information, such as the behaviour of y near $x = \pm\sqrt{3}$), we can produce a sketch graph of the function. Figure 28 exhibits all the main features.

Exercise 34

Consider the function

$$f(v) = \frac{v}{4 + 1.5v + 0.008v^2} \quad (v \geq 0).$$

(a) (i) Find any values of v for which $f(v)$ is not defined.

(ii) Find any values of v for which $f(v)$ is zero.

(iii) Indicate how $f(v)$ behaves as v becomes large.

(iv) Find any local maxima or minima of $f(v)$, and evaluate $f(v)$ at these points.

(b) Sketch a graph of $f(v)$, and use it to deduce the global maximum and global minimum of the function.

6 Integration

Integration arises in two different contexts. First, it 'reverses the process of differentiation'. Subsection 6.1 provides a reminder of this basic idea, and Subsection 6.2 discusses how to find relatively simple integrals, referring to the Handbook for standard results if necessary. Subsection 6.3 looks at two special techniques for finding more complicated integrals by hand.

Integration also arises as a kind of summation. For example, the mass of an object can be expressed as the integral of a function that describes how the object's density varies from point to point. Such *definite integrals* are discussed in Subsection 6.4.

6.1 Reversing differentiation

Diagnostic test

Try Exercise 35. If your answer agrees with the solution, you may proceed quickly to Subsection 6.2.

In the rest of this book, and elsewhere in the module, you will meet a variety of *differential equations*. These are equations involving the derivative of a function.

For example, if x is the position of a particle and t is time, then the velocity ($v = dx/dt$) might be given by the equation

$$\frac{dx}{dt} = 5t + 7. \tag{72}$$

The objective is usually to 'solve' the equation, so in this case we want to find the position x as a function of time t. To do this involves 'reversing' the differentiation, and this process is referred to as **integration**. In the above example we **integrate** both sides of the equation with respect to t, obtaining

$$x = \int (5t + 7)\,dt. \tag{73}$$

To evaluate this integral, you can use a table of standard integrals in the Handbook. The result is

$$x = \tfrac{5}{2}t^2 + 7t + C, \tag{74}$$

where C can be any constant. To confirm that equation (74) really is the solution of equation (72), we can differentiate it, obtaining

$$\frac{dx}{dt} = \frac{d}{dt}(\tfrac{5}{2}t^2 + 7t + C) = 5t + 7,$$

as required. The constant C is often referred to as an **arbitrary constant** or a **constant of integration**. Its presence means that the differential equation (72) does not have a *unique* solution.

Note that the arbitrary constant is included once the expression has been integrated (as in equation (74)), not before.

More generally, suppose that $f(x)$ is a known function, and $F(x)$ is an unknown function satisfying the differential equation

$$F'(x) = f(x).$$

Then we write the solution of this equation as

$$F(x) = \int f(x)\,dx,$$

where the right-hand side, $\int f(x)\,dx$, is called the **indefinite integral** of $f(x)$, and the function to be integrated, $f(x)$, is called the **integrand**.

For example, the differential equation

$$F'(x) = \frac{1}{1+x^2} \tag{75}$$

has solution

$$F(x) = \int \frac{1}{1+x^2}\,dx.$$

We now need to find the integral. In this case we are lucky, since the table of standard derivatives in the Handbook gives

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Any function that differentiates to $1/(1+x^2)$, such as $F(x) = \arctan x + 5$, is referred to as an **integral** (or *antiderivative*) of $1/(1+x^2)$.

Hence we have

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C,$$

where C is an arbitrary constant.

By contrast, consider finding the integral

$$\int \exp(-x^2)\,dx.$$

At first sight, this might seem no harder a problem to solve than equation (75). In fact, however, it is impossible! To be more precise, there is no simple combination of the elementary functions (polynomials, sin, cos, exp and ln) that when differentiated gives $\exp(-x^2)$.

Finding explicit expressions for integrals is a much harder task than finding derivatives. The rules of differentiation ensure that we can, in principle, find an explicit expression for the derivative of any combination of elementary functions. The equivalent is not true for integrals. What is more, even when integrals can be found, the working needed can be messy. The art of integration often relies on recognising patterns and knowing what works in particular cases.

There is a table of standard integrals in the Handbook. An integral will be readily found if it fits into one of the patterns listed there (such as $\int e^{ax}\,dx$ or $\int x^a\,dx$ for $a \neq -1$). Note that the table in the Handbook does not include arbitrary constants – you must supply these yourself. There are also simple rules for integrating constant multiples and sums. More complicated integrals can often be found using two techniques that will be introduced later: *integration by substitution* and *integration by parts*.

The table of standard integrals in the Handbook contains quite a wide selection of integrals. Some of these integrals are deduced from the table of standard derivatives, others by using integration by parts or substitution. You can regard them all as the fruit of others' experience, and draw on them as needed. The correctness of an integral obtained by using the Handbook can always be verified by differentiation.

Exercise 35

Use differentiation to verify that the following integrals are correct (where $a \neq 0$ is a constant and C is an arbitrary constant).

(a) $\displaystyle\int x\sin(ax)\,dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) + C$

(b) $\displaystyle\int \tan(ax)\,dx = -\frac{1}{a}\ln(\cos(ax)) + C \quad \left(-\tfrac{\pi}{2} < ax < \tfrac{\pi}{2}\right)$

6.2 Evaluating integrals

Diagnostic test

Try Exercises 37(a), 37(b) and 39. If your answers agree with the solutions, you may proceed quickly to Subsection 6.3.

Your first recourse for finding an integral by hand is the table of standard integrals in the Handbook. If the integrals of functions f and g are known, then the integral of $af + bg$, where a and b are constants, is readily found, using the rule

Or, preferably, use your memory!

$$\int \bigl(a\, f(x) + b\, g(x)\bigr)\, dx = a \int f(x)\, dx + b \int g(x)\, dx. \tag{76}$$

So, for example (referring to the Handbook for $\int e^{2x}\, dx$ and $\int x^7\, dx$),

$$\begin{aligned}\int (4e^{2x} + 9x^7)\, dx &= 4\int e^{2x}\, dx + 9\int x^7\, dx \\ &= 4\bigl(\tfrac{1}{2}e^{2x}\bigr) + 9\bigl(\tfrac{1}{8}x^8\bigr) + C \\ &= 2e^{2x} + \tfrac{9}{8}x^8 + C.\end{aligned}$$

Each of the integrals introduces an arbitrary constant, but the sum of these is also an arbitrary constant, so we need only one.

Sometimes algebraic manipulation can transform an expression to be integrated into a more amenable form. For example, the manipulation

$$\frac{3x^2 + 2x}{\sqrt{x}} = \frac{3x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} = 3x^{3/2} + 2x^{1/2}$$

transforms the expression on the left into a sum of constant multiples of integrals tabulated in the Handbook. Less obvious transformations can be achieved using trigonometric formulas. For example, equation (42) can be rearranged to give

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$

which enables us to integrate $\cos^2 x$.

Exercise 36

Use the identity

$$\cos^2(ax) = \tfrac{1}{2}[1 + \cos(2ax)]$$

(where $a \neq 0$ is a constant) to obtain $\displaystyle\int \cos^2(ax)\, dx$.

At times, attention needs to be paid to domains, to avoid giving, as integrals, expressions that are not defined. For example, if $x > 0$, then there is no difficulty in writing

$$\int \frac{1}{x}\, dx = \ln x + C \quad (x > 0), \tag{77}$$

but the right-hand side of this expression makes no sense if $x < 0$ because $\ln x$ is not defined in this case. (The domain of $\ln x$ is $x > 0$.) For $x < 0$ we have instead

$$\int \frac{1}{x}\,dx = \ln(-x) + C \quad (x < 0). \tag{78}$$

Equations (77) and (78) are both valid, because

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \quad\text{and}\quad \frac{d}{dx}\ln(-x) = \frac{1}{-x}\times(-1) = \frac{1}{x}.$$

The results in equations (77) and (78) are sometimes combined in the formula

$$\int \frac{1}{x}\,dx = \ln(|x|) + C \quad (x \neq 0),$$

where $|x|$ is the modulus of x, equal to $+x$ if $x \geq 0$, and to $-x$ if $x < 0$.

Exercise 37

Use the standard integrals given in the Handbook as necessary.

Find the following integrals.

(a) $\displaystyle\int e^{5x}\,dx$ (b) $\displaystyle\int 6\sec^2(3t)\,dt$ (c) $\displaystyle\int \frac{1}{36+4v^2}\,dv$

(d) $\displaystyle\int \frac{1}{3-2y}\,dy \quad (y < \tfrac{3}{2})$ (e) $\displaystyle\int \frac{1}{3-2y}\,dy \quad (y > \tfrac{3}{2})$

Exercise 38

Use the standard integrals given in the Handbook as necessary.

Find the following integrals.

(a) $\displaystyle\int (6\cos(-2t) + 8\sin(4t))\,dt$ (b) $\displaystyle\int \frac{1}{\sqrt{9-t^2}}\,dt \quad (-3 < t < 3)$

(c) $\displaystyle\int \frac{5t^3+7}{t}\,dt \quad (t < 0)$ (d) $\displaystyle\int \left(2\ln(4t) - \frac{2}{t}\right)dt \quad (t > 0)$

(e) $\displaystyle\int \frac{1}{(x-1)(x+1)}\,dx \quad (-1 < x < 1)$

The next example again uses a standard integral from the Handbook, but requires careful matching of parameters and attention to domains.

Example 6

For $x > \dfrac{1}{A} > 0$, find $I = \displaystyle\int \frac{1}{x(1-Ax)}\,dx$.

Solution

The Handbook contains the standard integral

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C.$$

The given integral I can be expressed in the form

$$I = \frac{-1}{A(x-0)\left(x-\frac{1}{A}\right)},$$

which matches the Handbook integral with $a = 1/A$ and $b = 0$.

Hence

$$\begin{aligned} I = -\frac{1}{A}\int \frac{1}{(x-0)\left(x-\frac{1}{A}\right)}\,dx &= -\frac{1}{A}\left(\frac{1}{\frac{1}{A}-0}\ln\left|\frac{x-\frac{1}{A}}{x-0}\right|\right) + C \\ &= -\ln\left|\frac{x-\frac{1}{A}}{x}\right| + C. \end{aligned}$$

The modulus signs can be removed because $x > 1/A$ and $x > 0$. Hence

$$I = \ln\left(\frac{x}{x-\frac{1}{A}}\right) + C = \ln\left(\frac{Ax}{Ax-1}\right) + C.$$

Exercise 39

For $k > 0$ and $-k < v < k$, find $\displaystyle\int \frac{1}{v^2-k^2}\,dv$.

(*Hint*: Remember that $v^2 - k^2 = (v-k)(v+k)$.)

6.3 Integration by substitution and by parts

Diagnostic test

Try Exercises 40(a), 40(b) and 42(a). If your answers agree with the solutions, you may proceed quickly to Subsection 6.4.

This subsection looks at two useful methods for finding more complicated integrals. Deciding which method to use in any particular case comes through experience.

Integration by substitution

Many integrals are of the form

$$\int f(u(x))\,\frac{du}{dx}\,dx,$$

or are constant multiples of such an expression. For example, the integral

$$\int \cos(2+3x^2)\times 6x\,dx \tag{79}$$

is of this form, with

$$u(x) = 2+3x^2, \quad f(u) = \cos(u) \quad \text{and} \quad \frac{du}{dx} = 6x.$$

In this case, $f(u(x)) = \cos(2 + 3x^2)$ is the composition of an 'inner function' $u(x) = 2 + 3x^2$ and an 'outer function' $f(u) = \cos(u)$. But $f(u(x))$ is only part of the integrand because it is multiplied by the derivative $du/dx = 6x$ of the inner function, $u(x) = 2 + 3x^2$.

This method is also referred to as **integration by change of variable**.

Integrals like this can be found using the method of **integration by substitution**, based on the following formula.

Formula for integration by substitution

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du, \tag{80}$$

or, in Leibniz notation,

$$\int f(u(x))\, \frac{du}{dx}\, dx = \int f(u)\, du. \tag{81}$$

In equation (81), the expression $\dfrac{du}{dx}\, dx$ in the integral on the left is effectively replaced by du in the integral on the right. This is worth remembering, and will be used later on.

Example 7

Find $\displaystyle\int 6x \cos(2 + 3x^2)\, dx$.

Solution

We try the substitution $u(x) = 2 + 3x^2$. Then $f(u) = \cos(u)$ and $du/dx = 6x$. So

$$\int \cos(2 + 3x^2)\, 6x\, dx = \int \cos(u)\, \frac{du}{dx}\, dx = \int \cos(u)\, du.$$

The integral over u is now easy. It is $\sin(u) + C$, so we have

$$\int \cos(2 + 3x^2)\, 6x\, dx = \sin(2 + 3x^2) + C,$$

where we have substituted for u in terms of x.

The rule for integration by substitution can be thought of as the reverse of the rule for differentiating a composite function.

The answer can be checked by differentiation:

$$\frac{d}{dx}(\sin(2 + 3x^2) + C) = \cos(2 + 3x^2) \times 6x.$$

Sometimes a rearrangement is needed to coax the integrand into a suitable form. For example, given the integral $\int x \cos(2 + 3x^2)\, dx$, we need to recognise that this is essentially the same as the integral that we have just considered, but with $6x$ replaced by x. Such a constant multiple is easily dealt with by writing

$$\int x \cos(2 + 3x^2)\, dx = \tfrac{1}{6} \int \cos(2 + 3x^2) \times 6x\, dx.$$

We then proceed just as before to obtain $\frac{1}{6}\sin(2+3x^2)+C$.

The key to using this method is recognising when the integrand has a suitable form: be on the lookout for an inner function $u(x)$ whose derivative du/dx appears as a multiplicative factor in the integrand.

One form of integral comes up sufficiently often to be worth special mention. If $g(x) \neq 0$, integration by substitution gives

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C. \tag{82}$$

To see why this is so, make the substitution $u = g(x)$. Then $du/dx = g'(x)$, so

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{1}{u}\frac{du}{dx}\,dx = \int \frac{1}{u}\,du.$$

Finding the integral of $1/u$ requires some care with domains (see the text following Exercise 36). For $u > 0$ it is $\ln(u) + C$, and for $u < 0$ it is $\ln(-u) + C$. These cases correspond to $g(x) > 0$ and $g(x) < 0$, respectively. Both cases are included by using the modulus sign in equation (82).

Exercise 40

Find the following integrals by making suitable substitutions.

(a) $\displaystyle\int y^2\exp(2+4y^3)\,dy$ (b) $\displaystyle\int \cos y\sin^2 y\,dy$

(c) $\displaystyle\int t\sqrt{1-t^2}\,dt \quad (-1<t<1)$ (d) $\displaystyle\int \frac{x}{1+x^2}\,dx$

(e) $\displaystyle\int \frac{\sin 2t}{1+\sin^2 t}\,dt$ (f) $\displaystyle\int \frac{y}{1-y^2}\,dy \quad (y \neq \pm 1)$

One advantage of knowing your derivatives is that you can adjust your tactics, choosing $u(x)$ so that $u'(x)$ appears where needed in the integrand.

The identity $\sin(2t) = 2\sin t\cos t$ may be useful in part (e).

All the integrals considered in Exercise 40 have integrands that can be expressed as a constant times $f(u(x))\,u'(x)$ for some choice of $u(x)$. We are not always so fortunate. The integral in the following example is not of this pattern, but we can still use a substitution to help find it.

Example 8

Find the integral $I = \displaystyle\int \frac{x^2}{\sqrt{2x-1}}\,dx$ for $x > 1/2$.

Solution

We try the substitution $u = 2x-1$. Then $du/dx = 2$ and the denominator in the integrand becomes $\sqrt{u} = u^{1/2}$. We still need to express the numerator x^2 in terms of u, and the element of integration dx in terms of du. The first task is easily achieved. Rearranging our equation for u, we get $x = (u+1)/2$, so $x^2 = (u+1)^2/4$.

To express dx in terms of du, we can use the fact noted earlier that within the integral in equation (81), we can effectively make the replacement $(du/dx)\,dx = du$. Since $du/dx = 2$, we have $2\,dx = du$, so we can make the replacement $dx = \frac{1}{2}\,du$.

Putting all this together, we get

$$I = \int \frac{x^2}{\sqrt{2x-1}}\,dx = \int \tfrac{1}{4}\,\frac{(u+1)^2}{u^{1/2}}\,\tfrac{1}{2}\,du.$$

Now we are in business! A straightforward calculation gives

$$\begin{aligned} I &= \tfrac{1}{8}\int \frac{u^2+2u+1}{u^{1/2}}\,du \\ &= \tfrac{1}{8}\int (u^{3/2} + 2u^{1/2} + u^{-1/2})\,du \\ &= \tfrac{1}{8}\left(\tfrac{2}{5}u^{5/2} + \tfrac{4}{3}u^{3/2} + 2u^{1/2}\right) + C \\ &= \tfrac{1}{20}(2x-1)^{5/2} + \tfrac{1}{6}(2x-1)^{3/2} + \tfrac{1}{4}(2x-1)^{1/2} + C. \end{aligned}$$

Exercise 41

Use the substitution $u = 3x+1$ to find $\displaystyle\int \frac{9x^2+1}{3x+1}\,dx$ for $x > -1/3$.

Integration by parts

The method of **integration by parts** is based on the following formula.

Formula for integration by parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx, \tag{83}$$

or, in Leibniz notation,

$$\int f(x)\,\frac{dg}{dx}\,dx = f(x)\,g(x) - \int \frac{df}{dx}\,g(x)\,dx. \tag{84}$$

Note that an arbitrary constant need not be included in the expression for $g(x)$ here, and it is usually omitted.

As with integration by substitution, this formula transforms an integral into a different one, and the key to success is to ensure that the 'new' integral is easier to evaluate than the original.

Example 9

Find $\displaystyle\int xe^{-2x}\,dx$.

Solution

Take $f(x) = x$ and $g'(x) = e^{-2x}$. Then $f'(x) = 1$ and $g(x) = -\frac{1}{2}e^{-2x}$, so

$$\begin{aligned}\int xe^{-2x}\,dx &= x\left(-\tfrac{1}{2}e^{-2x}\right) - \int 1\times\left(-\tfrac{1}{2}e^{-2x}\right)\,dx\\ &= -\tfrac{1}{2}xe^{-2x} + \tfrac{1}{2}\int e^{-2x}\,dx\\ &= -\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} + C\\ &= -\tfrac{1}{4}(2x+1)e^{-2x} + C.\end{aligned}$$

Our motive for splitting the integrand into $x \times e^{-2x}$ in this way, is that x becomes simpler when it is differentiated, while e^{-2x} at least gets no more complicated when it is integrated. So we end up with an integral that is easier to perform.

Exercise 42

(a) Use integration by parts to find $\displaystyle\int xe^{-x}\,dx$.

(b) Use integration by parts and the result of part (a) to find $\displaystyle\int x^2e^{-x}\,dx$.

(c) Use integration by parts to find $\displaystyle\int e^x \sin x\,dx$. You will need to integrate by parts twice, and compare your answer with the original integral.

6.4 Definite integrals

Diagnostic test

Try Exercises 43, 45 and 46. If your answers agree with the solutions, you may proceed quickly through this subsection.

An indefinite integral is a *function* (or, to be exact, a family of functions containing an arbitrary constant). A different, though closely related, integral is the **definite integral**, whose value is a *number*.

If $F(x)$ is the integral of $f(x)$, so that

$$\int f(x)\,dx = F(x),$$

then the definite integral of $f(x)$ between $x = a$ and $x = b$ is written as $\int_a^b f(x)\,dx$, and is defined by

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{85}$$

The value $x = a$ is called the **lower limit of integration**, and the value $x = b$ is called the **upper limit of integration**. A particular expression for $F(x)$ will contain an arbitrary constant, but this does not affect the value of the definite integral because the constant cancels out in equation (85). The difference $F(b) - F(a)$ is commonly written as $\big[F(x)\big]_a^b$ or $F(x)\big|_a^b$. So, for example,

$$\int_0^1 \frac{1}{\sqrt{4-\theta^2}}\,d\theta = \left[\arcsin\left(\tfrac{1}{2}\theta\right)\right]_0^1 = \arcsin\tfrac{1}{2} - \arcsin 0 = \tfrac{\pi}{6} - 0 = \tfrac{\pi}{6}.$$

A useful way of thinking of a definite integral $\int_a^b f(x)\,dx$ is as an accumulation of small quantities as x varies from a to b. For example, suppose that a straight rod lies along the x-axis and has a density per unit length given by the function $f(x)$, which varies along the length of the rod. The mass of a small segment of this rod, of length δx, centred on x, is approximated by $f(x)\,\delta x$. Some approximation is involved because the density varies inside the segment, but this variation is very slight if the segment is short enough. If we add up the masses of all the segments, we obtain the total mass of the rod. We can do this in the limit where the length of each segment approaches zero and the number of segments becomes huge. This improves the approximation mentioned above, and *in the limit* gives the *exact* total mass of the rod as the definite integral

$$M = \int_a^b f(x)\,dx,$$

where $x = a$ and $x = b$ mark the ends of the rod.

Example 10

A straight rod lies along the x-axis between $x = 2$ and $x = 4$ (measured in metres). The density of the rod per unit length depends on x, and is given by the function $f(x) = 3x^2 - x$ (measured in kilograms per metre). What is the total mass of the rod?

Solution

The total mass of the rod is

$$M = \int_2^4 f(x)\,dx = \int_2^4 (3x^2 - x)\,dx = \left[x^3 - \tfrac{1}{2}x^2\right]_2^4 = 56 - 6 = 50.$$

So the rod has mass 50 kilograms.

Exercise 43

Calculate each of the following definite integrals.

(a) $\displaystyle\int_0^1 (x^3 - 2)\,dx$

(b) $\displaystyle\int_1^2 (x^3 - 2)\,dx$

(c) $\displaystyle\int_0^2 (x^3 - 2)\,dx$

How is this integral related to those in parts (a) and (b)?

Exercise 44

Evaluate $\displaystyle\int_0^{3/2} \frac{1}{9+4z^2}\,dz.$

Definite integrals can be evaluated by finding the corresponding indefinite integral, and then using the limits of integration. When using integration by substitution, there is an extra step involving conversion of the limits.

Example 11

Suppose that we wish to evaluate the integral

$$I = \int_{1/2}^{3/2} \frac{1}{\sqrt{4-(2x-1)^2}}\,dx.$$

The closest integral to this one in the table of standard integrals in the Handbook is

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right).$$

To convert I into this form, we change the variable of integration to $u = 2x - 1$. Then $du/dx = 2$. This allows us to write $du = (du/dx)\,dx = 2\,dx$, so we can make the replacement $dx = \frac{1}{2}\,du$. Consequently, our integral becomes

$$I = \frac{1}{2}\int_{x=1/2}^{x=3/2} \frac{1}{\sqrt{4-u^2}}\,du. \tag{86}$$

Here the limits are written as $x = 1/2$ and $x = 3/2$ to explicitly show that they refer to the variable x rather than u. We can easily convert to the appropriate limits for u, using the relationship between u and x. If $x = 1/2$, then $u = 0$, and if $x = 3/2$, then $u = 2$. Hence, using the integral from the Handbook, the integral I becomes

$$I = \frac{1}{2}\int_{u=0}^{u=2} \frac{1}{\sqrt{4-u^2}}\,du = \frac{1}{2}\left[\arcsin\left(\frac{u}{2}\right)\right]_{u=0}^{u=2} = \frac{\pi}{4}.$$

Exercise 45

Suppose that we are told that for any positive integer n,

$$J_n = \int_0^{n\pi} u^2 \sin^2 u\,du = \frac{n^3\pi^3}{6} - \frac{n\pi}{4}.$$

Use this integral to evaluate

$$I = \int_0^1 x^2 \sin^2(3\pi x)\,dx.$$

Areas and the use of symmetry

Another way of thinking about a definite integral is in terms of an area. If $f(x) \geq 0$ for $a \leq x \leq b$, then the definite integral $\int_a^b f(x)\,dx$ is equal to the area under the graph of $f(x)$ between $x = a$ and $x = b$ (see Figure 29(a)). There is one point to be careful about here. If $f(x) < 0$, corresponding to a region *below* the x-axis, then we have a *negative* contribution to the integral, whereas area is always a *positive* quantity. Thus for a function f as pictured in Figure 29(b), $\int_a^b f(x)\,dx = \text{area } A_1 - \text{area } A_2$.

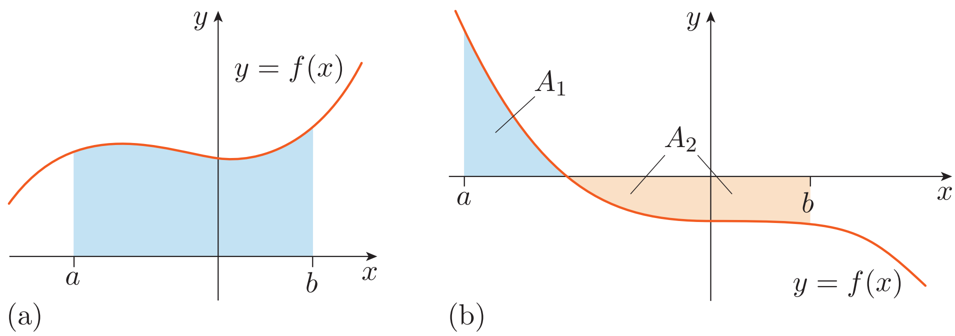


Figure 29 $\int_a^b f(x)\,dx$ as an area: (a) with $f(x) > 0$; (b) in general

It is possible for the positive contribution to an definite integral to exactly cancel the negative contribution, giving a zero result. This leads to a useful shortcut in some cases. Recall the following definitions:

- A function $f(x)$ is said to be **odd** if $f(-x) = -f(x)$ for all x.
- A function $f(x)$ is said to be **even** if $f(-x) = +f(x)$ for all x.

If a definite integral is over a range from $-a$ to a, where a is a constant, it is worth checking whether the integrand is an odd function, as you will now see.

Suppose that you are asked to calculate the following definite integrals:

$$I_1 = \int_{-\pi}^{\pi} \sin(x)\,dx, \quad I_2 = \int_{-2}^{2} \sin(x^3)\,dx.$$

You could set about evaluating these, using the methods that you have learned. And you might conclude that the second integral is a real challenge, because you don't know how to find an indefinite integral for $\sin(x^3)$. But there is an alternative approach that is very valuable. Figures 30 and 31 show plots of the integrands over their ranges of integration.

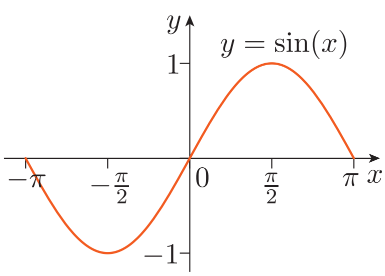


Figure 30 Graph of $\sin(x)$ over the interval $[-\pi, \pi]$

From Figure 30 we see that $\sin(x)$ is an odd function (that is, $\sin(-x) = -\sin(x)$). Because the limits of integration are equally spaced either side of the origin, the contribution to the definite integral coming from any positive value of x is exactly cancelled by a contribution from $-x$. By symmetry, we see that the integral must be equal to zero: $I_1 = 0$. This argument can be generalised as follows.

Definite integral of an odd function over a symmetric range

The integral of an odd function over a range $-a \leq x \leq a$ vanishes:

$$\int_{-a}^{a} f(x)\,dx = 0.$$

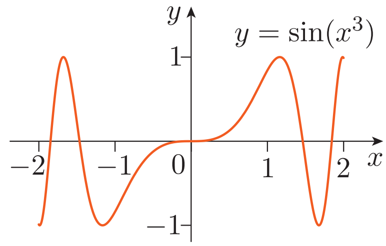


Figure 31 Graph of $\sin(x^3)$ over the interval $[-2, 2]$

Similarly, Figure 31 shows that $\sin(x^3)$ is an odd function. This can be confirmed by noticing that if $f(x) = \sin(x^3)$, then

$$f(-x) = \sin((-x)^3) = \sin(-x^3) = -\sin(x^3) = -f(x).$$

In I_2, $\sin(x^3)$ is integrated over a range centred on the origin, so we can immediately say that $I_2 = 0$ – in spite of being unable to find the corresponding indefinite integral!

Evaluating definite integrals can be tedious and a source of errors when performing calculations by hand. Looking out for opportunities to use symmetries to show that integrals are equal to zero, or else equal to others that you have already calculated, is a valuable skill that will save you time.

Exercise 46

Use symmetry to evaluate (or help to evaluate) the following integrals.

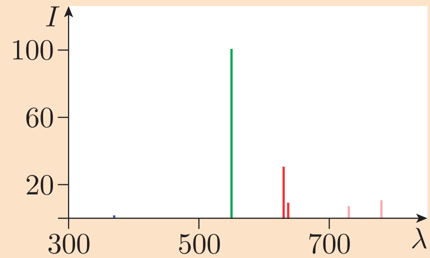
(a) $\displaystyle\int_{-\pi}^{\pi} e^{-x^2} \sin x\,dx$ (b) $\displaystyle\int_{-1}^{1} x \sin(1 + x^4)\,dx$

(c) $\displaystyle\int_{-1}^{1} (x^2 + x^3) \cos(x^3)\,dx$

Spectral lines and symmetry

Atoms emit light at certain definite frequencies (corresponding to 'spectral lines' of definite colours, e.g. Figure 32(a)). The frequencies fall into well-defined patterns, but with certain frequencies in the pattern missing. It turns out that the brightness of each predicted frequency is related to a definite integral. The missing frequencies correspond to integrals that are equal to zero, and this often happens for reasons of symmetry similar to those discussed above.

The fact that many frequencies are missing means that spectra are much simpler than they would otherwise be, and this gives us the chance to analyse complicated spectra, and find what mixture of substances is responsible for them. Such understanding is vital in fields as diverse as nuclear physics, astronomy, medicine and atmospheric science (see Figure 32(b)).



(a)

(b)

Figure 32 (a) A graph of intensity I against wavelength λ for the visible spectral lines emitted by oxygen atoms. These spectral lines are limited in number because certain integrals vanish for reasons of symmetry. The green line is by far the most intense, and this is responsible for the magnificent green glow of (b) the Aurora Borealis.

Learning outcomes

After studying this unit, you should be able to do the following.

- Understand the following terms: variable, dependent variable, independent variable, parameter; domain and image set of a function; inverse function, linear function, quadratic function.
- Solve two simultaneous linear equations by Gaussian elimination.
- Solve quadratic equations, using a formula or by factorisation.
- Compose functions and recognise compositions of functions.
- Recognise, sketch graphs of and manipulate the power, exponential and (natural) logarithm functions.
- Recognise, sketch graphs of and manipulate trigonometric functions and their inverses.
- Be aware of the series expansions of sin, cos and exp.
- Add, subtract, multiply and divide complex numbers, and move between Cartesian, polar and exponential forms of a complex number. Express sin and cos in terms of complex exponentials.
- Differentiate functions, using a table of standard derivatives and the rules for differentiating products, quotients and compositions of functions.
- Identify stationary points, local maxima and minima, and asymptotes; use such information to sketch graphs of functions.
- Find indefinite integrals, using a table of standard integrals and (in simple cases) rules for integration by substitution and by parts.
- Find definite integrals, using substitution and symmetry arguments where appropriate.

Solutions to exercises

Solution to Exercise 1

(a) The domain of $f(x)$ is given in the question as $x \geq 0$. Since $f(0) = 6$, and $f(x)$ increases indefinitely as x increases, the image set of $y = f(x)$ is $y \geq 6$. This can also be written as $6 \leq y < \infty$.

(b) Setting $y = 2x^2 + 6$ and solving for x gives $x = \pm\sqrt{\frac{1}{2}(y-6)}$. However, we are told that $x \geq 0$, so we can ignore the negative solution. The inverse function is therefore $x = g(y) = \sqrt{\frac{1}{2}(y-6)}$.

The domain of the inverse function $g(y)$ is equal to the image set of the original function, and so is $y \geq 6$. The image set of the inverse function $x = g(y)$ is equal to the domain of the original function, and so is $x \geq 0$.

Solution to Exercise 2

(a) For $Y = 2000$ at $t = -3600$, we have

$$2000 = 5(-3600) + c = -18\,000 + c.$$

Hence $c = 2000 + 18\,000 = 20\,000$.

(b) (i) The coastguard vessel catches the smuggler's boat when $X = Y$, i.e. when

$$7t = 5t + 20\,000.$$

This gives $2t = 20\,000$, so $t = 10\,000$.

10 000 seconds is 2 hours, 46 minutes and 40 seconds. So the smuggler's boat is caught at about 2.47 am.

(ii) At $t = 10\,000$, both X and Y are equal to $70\,000$. So the smuggler's boat is caught 70 km from A, which *is* inside territorial waters.

Solution to Exercise 3

Multiplying the first equation by $\frac{3}{2}$ gives

$$3u - \tfrac{15}{2}v = \tfrac{57}{2}.$$

Subtracting this from the second equation gives

$$\left(4 + \tfrac{15}{2}\right) v = -29 - \tfrac{57}{2},$$

i.e. $\frac{23}{2}v = -\frac{115}{2}$, so $v = -\frac{115}{23} = -5$.

Substituting this into the first equation gives

$$2u - 5(-5) = 19,$$

so $u = (19 - 25)/2 = -3$.

Thus the solution is $u = -3$, $v = -5$.

(It is good practice to check solutions where you can, and this is easily done here. With $u = -3$ and $v = -5$, we have

$$2u - 5v = 2(-3) - 5(-5) = -6 + 25 = 19,$$
$$3u + 4v = 3(-3) + 4(-5) = -9 - 20 = -29,$$

so these values of u and v do satisfy the given equations.)

Solution to Exercise 4

(a) Using equation (9), we obtain

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times (-4)}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{49 + 32}}{4} \\ &= \frac{-7 \pm 9}{4} = \tfrac{1}{2} \text{ or } -4. \end{aligned}$$

(These solutions can be checked by substitution into the quadratic equation. For example, with $x = -4$,

$$2x^2 + 7x - 4 = 32 - 28 - 4 = 0,$$

as required.)

(b) Using equation (9), we obtain

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-6)}}{2 \times 1} \\ &= \frac{-1 \pm \sqrt{1 + 24}}{2} \\ &= \frac{-1 \pm 5}{2} = -3 \text{ or } 2. \end{aligned}$$

Solution to Exercise 5

Using equation (9), we obtain

$$\begin{aligned} x &= \frac{-2K \pm \sqrt{4K^2 - 4\omega^2}}{2} \\ &= \frac{-2K \pm 2\sqrt{K^2 - \omega^2}}{2} \\ &= -K \pm \sqrt{K^2 - \omega^2}, \end{aligned}$$

as required.

Solution to Exercise 6

(a) We could use equation (9); however, in these cases it is easier to factorise by hand.

This is a difference of two squares:

$$x^2 - a = (x - \sqrt{a})(x + \sqrt{a}).$$

(b) This can also be expressed in terms of a difference of two squares:

$$2x^2 - 8a = 2(x^2 - 4a) = 2(x - 2\sqrt{a})(x + 2\sqrt{a}).$$

(c) This is a perfect square. Let $y = x^2$. Then

$$x^4 - 6x^2 + 9 = y^2 - 6y + 9 = (y-3)^2 = (x^2-3)^2.$$

Solution to Exercise 7

(a) $a^3 a^5 = a^{3+5} = a^8$.

(b) $a^3/a^5 = a^{3-5} = a^{-2}$ (or $1/a^2$).

(c) $(a^3)^5 = a^{3\times 5} = a^{15}$.

(d) $(2^{-1})^4 \times 4^3 = 2^{-4} \times (2^2)^3 = 2^{-4} \times 2^6 = 2^2 = 4$.

(e) $8^{-1/3} = 1/8^{1/3} = 1/\sqrt[3]{8} = \frac{1}{2}$.

(f) $16^{3/4} = (16^{1/4})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$.

(g) $\left(\frac{4}{9}\right)^{3/2} = \left(\sqrt{\frac{4}{9}}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$.

(h) $(16x^4)^{1/2} = 16^{1/2}(x^4)^{1/2} = \sqrt{16}x^{4\times 1/2} = 4x^2$.

Solution to Exercise 8

(a) With $f(x) = x^2$ and $g(x) = 1/(x-1)$ we have the following.

(i) $f(g(x)) = (1/(x-1))^2 = (1/(x-1)^2)$.

(ii) $g(f(x)) = 1/(x^2 - 1)$.

Note that $g(f(x))$ is not the same as $f(g(x))$.

(b) We can obtain $h(x)$ in three steps.

Step 1 Apply the square function to x.

Step 2 Add 1 to the result of Step 1.

Step 3 Apply the sine function to the result of Step 2.

Then $h(x) = r(q(p(x)))$, where

$$p(x) = x^2, \quad q(x) = x + 1, \quad r(x) = \sin(x).$$

Solution to Exercise 9

For $x = 1/2$ the sum is

$$1 + \tfrac{1}{2} + \tfrac{(1/2)^2}{2} + \tfrac{(1/2)^3}{6} + \tfrac{(1/2)^4}{24} + \tfrac{(1/2)^5}{120} \simeq 1.649.$$

For $x = 1$ the sum is

$$1 + 1 + \tfrac{1^2}{2} + \tfrac{1^3}{6} + \tfrac{1^4}{24} + \tfrac{1^5}{120} + \tfrac{1^6}{720} \simeq 2.718.$$

For $x = 2$ the sum is

$$1 + 2 + \tfrac{2^2}{2} + \tfrac{2^3}{6} + \tfrac{2^4}{24} + \tfrac{2^5}{120} + \tfrac{2^6}{720} + \tfrac{2^7}{5040} \simeq 7.381.$$

These answers can be compared with those obtained by entering exp(1/2), exp(1) and exp(2) directly into a calculator:

$$\exp(1/2) = 1.648\,721\,271\ldots,$$
$$\exp(1) = 2.718\,281\,828\ldots,$$
$$\exp(2) = 7.389\,056\,099\ldots.$$

Our first two answers are correct to four significant figures. The last answer is correct to only two significant figures, but it could be made as accurate as we wish by adding more terms in the series. Note that as x increases, we need more terms in the sum to get an accurate answer for $\exp(x)$.

Solution to Exercise 10

Using a calculator we find

$$\begin{aligned}\exp(1/2)\times\exp(1/2) &= 1.648\,721\,271\ldots\times 1.648\,721\,271\ldots\\ &= 2.718\,281\,829\ldots,\end{aligned}$$

which agrees with $\exp(1) = 2.718\,281\,828$. (Any discrepancy in the last decimal place is insignificant because of rounding.)

Similarly,

$$\begin{aligned}\exp(1)\times\exp(1) &= 2.718\,281\,828\ldots\times 2.718\,281\,828\ldots\\ &= 7.389\,056\,096\ldots,\end{aligned}$$

which agrees with $\exp(2) = 7.389\,056\,099\ldots$.

Solution to Exercise 11

(a) $\ln 7 + \ln 4 - \ln 14 = \ln(7\times 4/14) = \ln 2$.

(b) $$\begin{aligned}\ln a + 2\ln b - \ln(a^2b) &= \ln a + \ln(b^2) - \ln(a^2b)\\ &= \ln\bigl(a\times b^2 \div (a^2b)\bigr)\\ &= \ln(b/a) \quad (\text{or } \ln b - \ln a).\end{aligned}$$

(c) To simplify $\ln(e^x\times e^y)$, we first rearrange it in the form $\ln(e^{something})$, which just equals *something*:

$$\ln(e^x\times e^y) = \ln(e^{x+y}) = x+y.$$

An alternative argument gives

$$\ln(e^x\times e^y) = \ln(e^x) + \ln(e^y) = x+y.$$

(d) In parts (d)–(f), we first rearrange the expression as $e^{\ln(something)}$, which also just equals *something*.

Here, $e^{2\ln x} = e^{\ln(x^2)} = x^2$.

(e) $e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2}$ (or $1/x^2$).

(f) $\exp\bigl(2\ln x + \ln(x+1)\bigr) = \exp\bigl(\ln(x^2\times(x+1))\bigr) = x^2(x+1)$.

An alternative argument gives

$$\exp\bigl(2\ln x + \ln(x+1)\bigr) = \exp\bigl(\ln(x^2)\bigr)\times\exp\bigl(\ln(x+1)\bigr) = x^2(x+1).$$

Solution to Exercise 12

Using equation (30) we have

$$A = e^{\pi \ln 5} = e^{5.0562} = 157.0$$

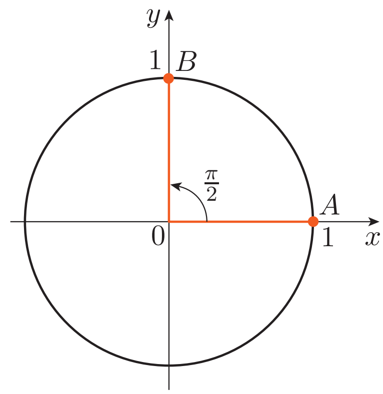
and

$$B = e^{-4.315 \ln 10} = e^{-9.9357} = 4.842 \times 10^{-5},$$

to four significant figures.

Solution to Exercise 13

(a) Note that the question asks us to use Figure 15, not read off values from Figure 16! The figure given here shows a circle of radius 1. The point A corresponds to a rotation through 0, and the point B corresponds to a rotation through $\frac{\pi}{2}$.



We see from the figure that A has coordinates $(1, 0) = (\cos 0, \sin 0)$, and B has coordinates $(0, 1) = \left(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}\right)$. Therefore

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \sin \tfrac{\pi}{2} = 1, \quad \cos \tfrac{\pi}{2} = 0.$$

(b) Since $\sin 0 = 0$, $\operatorname{cosec} 0$ and $\cot 0$ are not defined. We have

$$\tan 0 = 0 \quad \text{and} \quad \sec 0 = 1.$$

Since $\cos \frac{\pi}{2} = 0$, $\tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are not defined. We have

$$\cot \tfrac{\pi}{2} = 0 \quad \text{and} \quad \operatorname{cosec} \tfrac{\pi}{2} = 1.$$

(c) Using the triangles in Figure 17, we obtain

$$\begin{aligned}
&\sin \tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}, && \sin \tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}, && \sin \tfrac{\pi}{6} = \tfrac{1}{2},\\
&\cos \tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}, && \cos \tfrac{\pi}{3} = \tfrac{1}{2}, && \cos \tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2},\\
&\tan \tfrac{\pi}{4} = 1, && \tan \tfrac{\pi}{3} = \sqrt{3}, && \tan \tfrac{\pi}{6} = \tfrac{1}{\sqrt{3}},\\
&\cot \tfrac{\pi}{4} = 1, && \cot \tfrac{\pi}{3} = \tfrac{1}{\sqrt{3}}, && \cot \tfrac{\pi}{6} = \sqrt{3}.
\end{aligned}$$

(d) From Figure 16, $\sin \theta = 0$ for $\theta = 0$ and $\theta = \pi$, and for any value of θ that differs from 0 or π by any integer multiple of 2π. Hence we conclude that $\sin \theta = 0$ if $\theta = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$ (i.e. n is any integer). This can be written as $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers.

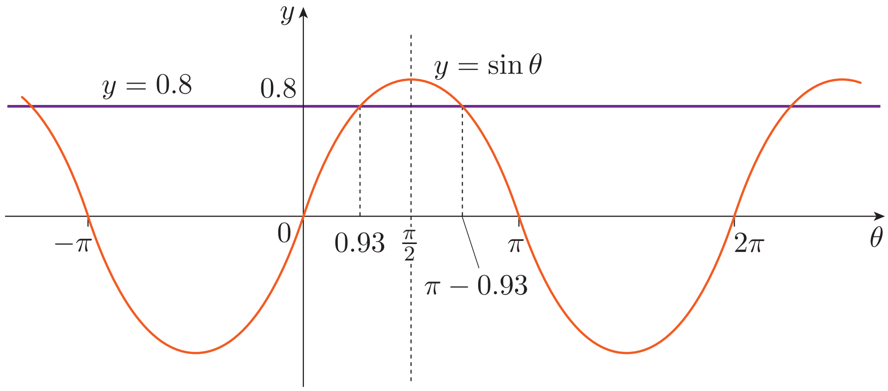
The symbol $\in$ means 'is a member of'.

Solution to Exercise 14

(a) Using a calculator, one solution is $\arcsin 0.8 = 0.93$ (to two decimal places). Looking at the graph of $y = \sin \theta$ shown below, we see that it is symmetric about $\theta = \frac{\pi}{2}$. So there is also a solution of $\sin \theta = 0.8$ at $\theta = \pi - 0.93 = 2.21$ (to two decimal places). These are the only solutions with θ between 0 and 2π. (If θ is between π and 2π, then $\sin \theta$ is negative.)

The other solutions are obtained by adding multiples of 2π to these two. The solutions in the required range are (to two decimal places)

$$0.93,\ 2.21,\ 7.21,\ 8.50,\ 13.49,\ 14.78.$$



(b) We saw in Exercise 13(c) that $\tan\frac{\pi}{4} = 1$, so $\theta = \frac{\pi}{4}$ is one solution. We can see from the graph of tan (Figure 18) that there is one solution of this equation in the range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and that other solutions are obtained by adding multiples of π to this. We can express the full set of solutions as

$$\tfrac{\pi}{4} + n\pi,$$

where n is any integer (positive, zero or negative).

Solution to Exercise 15

(a) Dividing equation (32) by $\cos^2\theta$ gives equation (35).

(b) Dividing equation (33) by equation (34) gives

$$\tan(\theta \pm \phi) = \frac{\sin\theta\cos\phi \pm \cos\theta\sin\phi}{\cos\theta\cos\phi \mp \sin\theta\sin\phi}.$$

Then dividing each term in the numerator and denominator of the right-hand side by $\cos\theta\cos\phi$ gives equation (37).

(c) Using equation (34) we obtain

$$\cos(\theta - \phi) - \cos(\theta + \phi) = 2\sin\theta\sin\phi,$$

which establishes the result in equation (38).

(d) Setting $\phi = \theta$ in equation (33) and taking the upper signs gives equation (41).

Solution to Exercise 16

(a)
$$\begin{aligned}\sin(2\pi - \theta) &= \sin 2\pi\cos\theta - \cos 2\pi\sin\theta \\ &= 0 \times \cos\theta - 1 \times \sin\theta = -\sin\theta.\end{aligned}$$

(b)
$$\begin{aligned}\sin\left(\tfrac{\pi}{2} - \theta\right) &= \sin\tfrac{\pi}{2}\cos\theta - \cos\tfrac{\pi}{2}\sin\theta \\ &= 1 \times \cos\theta - 0 \times \sin\theta = \cos\theta.\end{aligned}$$

For $0 < \theta < \frac{\pi}{2}$, this result can be confirmed by examination of a right-angled triangle.

(c) $$\begin{aligned}\sin(\pi-\theta) &= \sin\pi\cos\theta - \cos\pi\sin\theta \\ &= 0\times\cos\theta - (-1)\times\sin\theta = \sin\theta.\end{aligned}$$

(d) $$\begin{aligned}\cos(\pi-\theta) &= \cos\pi\cos\theta + \sin\pi\sin\theta \\ &= (-1)\times\cos\theta + 0\times\sin\theta = -\cos\theta.\end{aligned}$$

(e) $$\begin{aligned}\cos(2\pi-\theta) &= \cos 2\pi\cos\theta + \sin 2\pi\sin\theta \\ &= 1\times\cos\theta + 0\times\sin\theta = \cos\theta.\end{aligned}$$

(f) $$\begin{aligned}\cos\left(\tfrac{\pi}{2}-\theta\right) &= \cos\tfrac{\pi}{2}\cos\theta + \sin\tfrac{\pi}{2}\sin\theta \\ &= 0\times\cos\theta + 1\times\sin\theta = \sin\theta.\end{aligned}$$

For $0 < \theta < \frac{\pi}{2}$, this result can be confirmed by examination of a right-angled triangle.

(g) $$\begin{aligned}\cos\left(\tfrac{3\pi}{2}+x\right) &= \cos\tfrac{3\pi}{2}\cos x - \sin\tfrac{3\pi}{2}\sin x \\ &= 0\times\cos x - (-1)\times\sin x = \sin x.\end{aligned}$$

Solution to Exercise 17

(a) $\overline{v} = 3 + 4i$.

(b) $|v| = \sqrt{3^2+4^2} = 5$.

(c) $v - w = (3-4i) - (2-i) = 1 - 3i$.

(d) $vw = (3-4i)(2-i) = 6 - 8i - 3i + 4i^2 = 2 - 11i$.

(e) $$\begin{aligned}\frac{w}{v} = \frac{w\overline{v}}{|v|^2} &= \frac{(2-i)(3+4i)}{3^2+4^2} \\ &= \frac{6-3i+8i-4i^2}{25} \\ &= \tfrac{10}{25} + \tfrac{5}{25}i = \tfrac{2}{5} + \tfrac{1}{5}i.\end{aligned}$$

(f) $\dfrac{1}{w} = \dfrac{\overline{w}}{|w|^2} = \dfrac{2+i}{2^2+1^2} = \frac{2}{5} + \frac{1}{5}i.$

(g) $w^2 = (2-i)(2-i) = 4 - 2i - 2i + i^2 = 3 - 4i$.

(h) $2w - 3v = 4 - 2i - (9 - 12i) = -5 + 10i$.

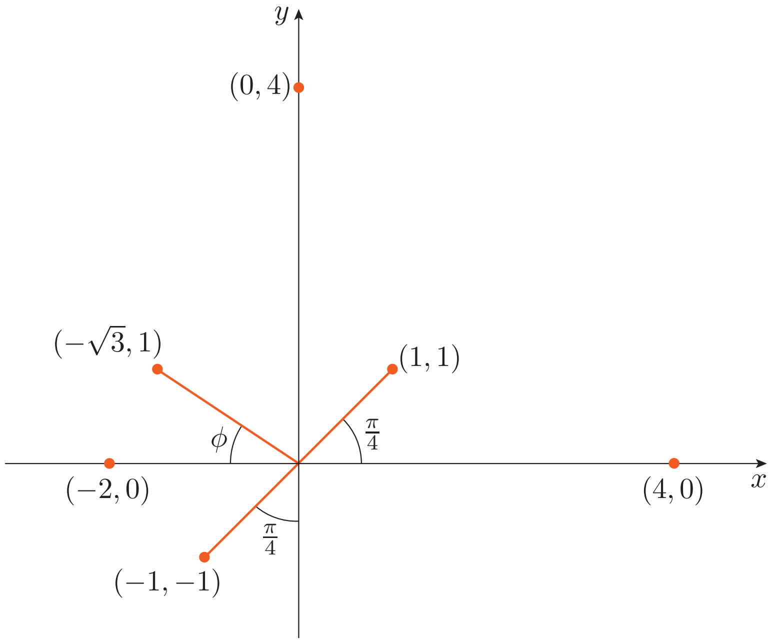
Solution to Exercise 18

We obtain

$$x = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm 2i}{4} = -\tfrac{1}{2} \pm \tfrac{1}{2}i.$$

Solution to Exercise 19

The points are illustrated in the figure below.



Using trigonometry in this diagram, the polar coordinates of these points are as follows:

$(x, y) = (-2, 0)$ corresponds to $(r, \theta) = (2, \pi)$,
$(x, y) = (1, 1)$ corresponds to $(r, \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$,
$(x, y) = (-1, -1)$ corresponds to $(r, \theta) = \left(\sqrt{2}, -\frac{3\pi}{4}\right)$,
$(x, y) = (4, 0)$ corresponds to $(r, \theta) = (4, 0)$,
$(x, y) = (0, 4)$ corresponds to $(r, \theta) = \left(4, \frac{\pi}{2}\right)$,
$(x, y) = (-\sqrt{3}, 1)$ corresponds to $(r, \theta) = (2, \pi - \phi)$,
where ϕ is as shown in the figure.

Now $\tan\phi = \frac{1}{\sqrt{3}}$, so $\phi = \frac{\pi}{6}$ (see the solution to Exercise 13(c)). So $(x, y) = (-\sqrt{3}, 1)$ has polar coordinates $(r, \theta) = \left(2, \frac{5\pi}{6}\right)$.

Solution to Exercise 20

$z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}(1 - i)$.

Solution to Exercise 21

(a) For all these numbers, we can use the diagram accompanying the solution to Exercise 19.

-2 has polar coordinates $(r, \theta) = (2, \pi)$. (Note that r must be positive, and the principal value of the argument must be in the range $-\pi < \theta \le \pi$.)

Writing the polar form in full, we get $-2 = 2(\cos\pi + i\sin\pi)$.

(b) $1+i$ has polar coordinates $(r,\theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$.

So $1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$.

(c) $-1-i$ has polar coordinates $(r,\theta) = \left(\sqrt{2}, -\frac{3\pi}{4}\right)$.

So $-1-i = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$.

(d) 4 has polar coordinates $(r,\theta) = (4,0)$.

So $4 = 4(\cos(0) + i\sin(0))$.

(e) $4i$ has polar coordinates $(r,\theta) = \left(4, \frac{\pi}{2}\right)$.

So $4i = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$.

(f) $-\sqrt{3}+i$ has polar coordinates $(r,\theta) = \left(\sqrt{3+1}, \pi - \frac{\pi}{6}\right) = \left(2, \frac{5\pi}{6}\right)$.

So $-\sqrt{3}+i = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$.

Solution to Exercise 22

(a) The modulus of z is 3, and the argument is $-7\pi/2$. We need to find the principal value of the argument, which means that we need to add some multiple of 2π to it so that it lies in the range $-\pi < \theta \leq \pi$. Clearly $-7\pi/2 + 4\pi = \pi/2$ lies in this range, so

$$z = 3e^{-i7\pi/2} = 3e^{i\pi/2}.$$

(b) The argument is 71π, which is equivalent to $71\pi - 70\pi = \pi$. So

$$z = e^{i71\pi} = e^{i\pi}.$$

Solution to Exercise 23

The exponential form of z is $z = re^{i\theta}$, so

$$ze^{i\omega t} = re^{i\theta}e^{i\omega t} = re^{i(\omega t+\theta)}.$$

Taking the real part of this expression, we obtain

$$\operatorname{Re}\left(ze^{i\omega t}\right) = \operatorname{Re}\left(re^{i(\omega t+\theta)}\right) = r\cos(\omega t+\theta).$$

Solution to Exercise 24

The polar coordinates of $z = 1-i$ are $(r,\theta) = \left(\sqrt{2}, -\frac{\pi}{4}\right)$, so the exponential form of z is

$$z = \sqrt{2}e^{-i\pi/4}.$$

Hence

$$\begin{aligned} z^{20} &= \left(\sqrt{2}e^{-i\pi/4}\right)^{20} \\ &= (\sqrt{2})^{20}e^{-i20\pi/4} \\ &= 2^{10}e^{-5i\pi} \\ &= 1024\,e^{i\pi}, \end{aligned}$$

where we have added 6π to the argument of the exponential to obtain the principal value in the last line. Returning to Cartesian form by using Euler's formula, we obtain

$$(1-i)^{20} = 1024(\cos\pi + i\sin\pi) = -1024.$$

Solution to Exercise 25

Using the Handbook result

$$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$$

with $a = 3$ and x replaced by t, we get

$$\dot{x} = \frac{dx}{dt} = -21\sin(3t).$$

To differentiate again, we use

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$

to obtain

$$\ddot{x} = \frac{d^2x}{dt^2} = -63\cos(3t).$$

Solution to Exercise 26

We need the Handbook result

$$\frac{d}{dx}(e^{ax}) = ae^{x}.$$

The rate at which the wage bill will be rising is

$$\frac{dB}{dt} = 10^5(0.04)\exp(0.04t) = 4000\exp(0.04t).$$

As a fraction of the future wage bill B, the rate of rise dB/dt is

$$\frac{1}{B}\frac{dB}{dt} = \frac{10^5(0.04)\exp(0.04t)}{10^5\exp(0.04t)} = 0.04.$$

So the rate of rise is 4% per year.

Solution to Exercise 27

(a) $\dfrac{dy}{dx} = (-9)(-5)\exp(-5x) = 45\exp(-5x).$

(b) $F'(x) = 12x^3 - 4$, so putting $x = 2$ gives

$$F'(2) = 12 \times 2^3 - 4 = 92.$$

(c) $\dfrac{dy}{dt} = \dfrac{1}{t}$, so $\dfrac{d^2y}{dt^2} = -\dfrac{1}{t^2}.$

(d) $g'(t) = -3a\sin(3t) + 3b\cos(3t)$, and then

$$g''(t) = -9a\cos(3t) - 9b\sin(3t),$$

so

$$g''(0) = -9a.$$

(e) $F'(x) = 6\sec(2x)\tan(2x) - 12\sin(-3x)$. Hence

$$\begin{aligned} F'\left(\tfrac{\pi}{6}\right) &= 6\sec\tfrac{\pi}{3}\tan\tfrac{\pi}{3} - 12\sin\left(-\tfrac{\pi}{2}\right) \\ &= 6 \times 2 \times \sqrt{3} - 12 \times (-1) = 12\sqrt{3} + 12. \end{aligned}$$

Solution to Exercise 28

Given $f(t) = \cos(2t) + i\sin(2t)$, we have $f'(t) = -2\sin(2t) + 2i\cos(2t)$ then

$$f''(t) = -4\cos(2t) - 4i\sin(2t).$$

Writing $f(t) = \exp(2it)$, we have $f'(t) = 2i\exp(2it)$ and

$$f''(t) = (2i)^2\exp(2it) = -4\exp(2it) = -4\cos(2t) - 4i\sin(2t),$$

as before.

Solution to Exercise 29

(a) The function $y = (\ln x)/(x^2+1)$ is a quotient. Using standard derivatives from the Handbook, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)(x^2+1) - (\ln x)(2x)}{(x^2+1)^2}\\ &= \frac{x + x^{-1} - 2x\ln x}{(x^2+1)^2}.\end{aligned}$$

This can be simplified by multiplying the numerator and denominator by x:

$$\frac{dy}{dx} = \frac{x^2 + 1 - 2x^2\ln x}{x(x^2+1)^2}.$$

(b) The function $f(t) = t^5\ln(3t)$ is a product. Recalling that

$$\frac{d}{dx}(\ln(ax)) = 1/x \quad \text{for } ax > 0,$$

we get

$$f'(t) = 5t^4\ln(3t) + t^5 \times \frac{1}{t} = 5t^4\ln(3t) + t^4.$$

(c) The function $g(t) = (At+B)\sin(Ct)$ is a product. We get

$$g'(t) = A\sin(Ct) + (At+B)C\cos(Ct),$$

so $g'(0) = BC$.

(d) Given $x(t) = e^{-3t}\sin(4t)$, we want to find $\dot{x}(t)$ and $\ddot{x}(t)$. Using the product rule:

$$\begin{aligned}\dot{x}(t) &= -3e^{-3t}\sin(4t) + e^{-3t}(4\cos(4t))\\ &= e^{-3t}(4\cos(4t) - 3\sin(4t)).\end{aligned}$$

Using the product rule again:

$$\begin{aligned}\ddot{x}(t) &= -3e^{-3t}(4\cos(4t) - 3\sin(4t)) + e^{-3t}(-16\sin(4t) - 12\cos(4t))\\ &= -e^{-3t}(7\sin(4t) + 24\cos(4t)).\end{aligned}$$

Solution to Exercise 30

(a) In each case, we indicate the split into inner and outer functions but (except in this part) omit details. It is perfectly acceptable to do the differentiation in your head, using the product of the derivatives of outer and inner functions.

Taking the inner function to be $u = t^2$, the outer function is $y(u) = \exp(u)$. We have $dy/du = \exp(u)$ and $du/dt = 2t$, so

$$\frac{dy}{dt} = \frac{dy}{du}\,\frac{du}{dt} = \exp(u)\,2t = 2t\exp(t^2).$$

(b) Taking the inner function to be $u = 3x^3 + 4$, the outer function is $f(u) = u^6$, so

$$f'(x) = 6(3x^3+4)^5 \times (9x^2) = 54x^2(3x^3+4)^5.$$

(c) Taking the inner function to be $u = 3v + 4$, the outer function is $z(u) = \tan u$, so

$$\frac{dz}{dv} = \sec^2(3v+4) \times 3 = 3\sec^2(3v+4).$$

(d) Taking the inner function to be $u = 4 - z^2$, the outer function is $g(u) = u^{1/2}$, so

$$g'(z) = \tfrac{1}{2}(4-z^2)^{-1/2} \times (-2z) = \frac{-z}{\sqrt{4-z^2}}.$$

(e) Taking the inner function to be $u = 1 + 2x^2$, the outer function is $f(u) = u^{-3/2}$ (see Example 2), so

$$f'(x) = -\tfrac{3}{2}\left(1+2x^2\right)^{-5/2} \times (4x) = \frac{-6x}{\left(\sqrt{1+2x^2}\,\right)^5}.$$

Solution to Exercise 31

(a) $\sec(x/(x^2+1))$ is a composite function, with

$$y = \sec u, \qquad u = \frac{x}{x^2+1}.$$

Here u is a quotient, and

$$\frac{du}{dx} = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Then, using the chain rule,

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} &= \sec u \tan u\,\frac{1-x^2}{(x^2+1)^2}\\ &= \frac{1-x^2}{(x^2+1)^2}\sec\left(\frac{x}{x^2+1}\right)\tan\left(\frac{x}{x^2+1}\right).\end{aligned}$$

(b) $t^2\exp(t^3+1)$ is a product of $u = t^2$ and $v = \exp(t^3+1)$. The second part of the product is a composite function with inner function $g = t^3 + 1$ and outer function $f = \exp g$. The chain rule gives

$$\frac{dv}{dt} = \exp(t^3+1) \times 3t^2 = 3t^2\exp(t^3+1).$$

Then, using the product rule,

$$\begin{aligned}\frac{dz}{dt} &= 2t\exp(t^3+1) + t^2(3t^2\exp(t^3+1)) \\ &= (3t^4+2t)\exp(t^3+1).\end{aligned}$$

Solution to Exercise 32

(a) (i) The product rule gives

$$\frac{d}{dx}(x^2y) = 2xy + x^2\,\frac{dy}{dx}.$$

(ii) The composite rule gives

$$\frac{d}{dx}(y^3) = 3y^2\,\frac{dy}{dx}.$$

(iii) The composite rule gives

$$\begin{aligned}\frac{d}{dx}(x+\sin(xy)) &= 1+\cos(xy)\,\frac{d}{dx}(xy) \\ &= 1+\cos(xy)\left(y + x\,\frac{dy}{dx}\right).\end{aligned}$$

(b) Using implicit differentiation, we obtain

$$3x^2 + 2xy + x^2\,\frac{dy}{dx} + 3y^2\,\frac{dy}{dx} = 0.$$

When $x=-1$ and $y=1$, this gives

$$3 - 2 + \frac{dy}{dx} + 3\,\frac{dy}{dx} = 0.$$

Hence $dy/dx = -\frac{1}{4}$, and the required gradient is $-\frac{1}{4}$.

Solution to Exercise 33

(a) To test for stationary points, use the product rule to find

$$\begin{aligned}y'(x) &= -2e^{-x/2} - (2(x+1))\times\left(-\tfrac{1}{2}e^{-x/2}\right) \\ &= (-2+(x+1))e^{-x/2} \\ &= (x-1)e^{-x/2}.\end{aligned}$$

So $y'(x)=0$ only at $x=1$. This is the only stationary point.

(b) To classify the stationary point, we differentiate again:

$$y''(x) = e^{-x/2} + (x-1)\left(-\tfrac{1}{2}\right)e^{-x/2} = \tfrac{1}{2}(3-x)e^{-x/2}.$$

At the stationary point, $y''(1) = e^{-1/2} > 0$, so the stationary point is a local minimum.

Alternatively, we see that the first derivative y' is negative if $x<1$ and positive if $x>1$, so the first derivative test confirms that it is a local minimum.

The value of the function at the stationary point is $y(1) = 5-4e^{-1/2} = 2.574$ (to four significant figures).

Solution to Exercise 34

(a) (i) The denominator $4 + 1.5v + 0.008v^2$ is positive for all $v \geq 0$, so $f(v)$ is defined for all $v \geq 0$.

(ii) $f(v) = 0$ only when $v = 0$.

(iii) As $v \to \infty$, $f(v) \to 0$.

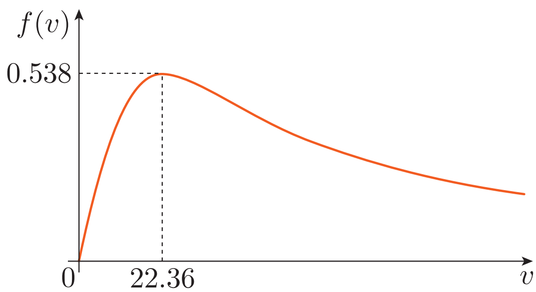
(iv) To find any stationary points, differentiate $f(v)$ using the quotient rule, to obtain

$$\begin{aligned} f'(v) &= \frac{(4 + 1.5v + 0.008v^2) - v(1.5 + 0.016v)}{(4 + 1.5v + 0.008v^2)^2} \\ &= \frac{4 - 0.008v^2}{(4 + 1.5v + 0.008v^2)^2}. \end{aligned}$$

The stationary points occur when $4 - 0.008v^2 = 0$, i.e. at $v = \pm\sqrt{500} = \pm 22.36$ (to two decimal places). The negative stationary point is outside the domain ($v \geq 0$), so we need consider only $v = 22.36$.

For $v < 22.36$, $f'(v) > 0$, while for $v > 22.36$, $f'(v) < 0$. Therefore $v = 22.36$ is a local maximum. At this point, $f(22.36) = 0.538$ (to three decimal places).

(b) A graph of f is shown below.



From the graph we see that the global maximum of f occurs at the local maximum, i.e. at $v = \sqrt{500} = 22.36$. The global minimum occurs at the endpoint of the domain, i.e. at $v = 0$.

Solution to Exercise 35

(a) Using the product rule for derivatives,

$$\begin{aligned} &\frac{d}{dx}\left(-\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) + C\right) \\ &\quad = -\frac{1}{a}\cos(ax) + \left(-\frac{x}{a}\right)[-a\sin(ax)] + \frac{a}{a^2}\cos(ax) \\ &\quad = x\sin(ax). \end{aligned}$$

Therefore

$$\int x\sin(ax)\,dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) + C,$$

so verifying the given integral.

(b) Using the composite rule for derivatives, and noting that $\cos(ax) > 0$ for $-\pi/2 < ax < \pi/2$, we have

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{a}\ln(\cos(ax)) + C\right) &= \left(-\frac{1}{a}\right)\frac{1}{\cos(ax)}\frac{d}{dx}(\cos(ax))\\ &= \left(-\frac{1}{a}\right)\frac{-a\sin(ax)}{\cos(ax)}\\ &= \tan(ax).\end{aligned}$$

Therefore

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln(\cos(ax)) + C,$$

so verifying the given integral.

Solution to Exercise 36

Using the given identity, we have

$$\begin{aligned}\int \cos^2(ax)\,dx = \frac{1}{2}\int [1+\cos(2ax)]\,dx &= \frac{1}{2}\left(x + \frac{1}{2a}\sin(2ax)\right) + C\\ &= \frac{1}{2}x + \frac{1}{4a}\sin(2ax) + C.\end{aligned}$$

Solution to Exercise 37

(a) $\displaystyle\int e^{5x}\,dx = \tfrac{1}{5}e^{5x} + C.$

(b) $\displaystyle\int 6\sec^2(3t)\,dt = 6 \times \tfrac{1}{3}\tan(3t) + C = 2\tan(3t) + C.$

(c) This does not quite match a Handbook integral as it stands. We have $36 + 4x^2 = 4(9 + x^2)$, so

$$\begin{aligned}\int \frac{1}{36+4v^2}\,dv = \int \frac{1}{4(9+v^2)}\,dv &= \tfrac{1}{4}\int \frac{1}{9+v^2}\,dv\\ &= \tfrac{1}{4}\left(\tfrac{1}{3}\arctan\left(\tfrac{1}{3}v\right)\right) + C\\ &= \tfrac{1}{12}\arctan\left(\tfrac{1}{3}v\right) + C.\end{aligned}$$

(d) For $y < \frac{3}{2}$, the integrand is positive, and we have

$$\int \frac{1}{3-2y}\,dy = -\tfrac{1}{2}\ln(3-2y) + C.$$

(e) For $y > \frac{3}{2}$, the integrand is negative, and we have

$$\int \frac{1}{3-2y}\,dy = -\tfrac{1}{2}\ln(-3+2y) + C = -\tfrac{1}{2}\ln(2y-3) + C.$$

Solution to Exercise 38

(a) $$\int (6\cos(-2t) + 8\sin(4t)) = -3\sin(-2t) - 2\cos(4t) + C.$$

Note that you can use $\cos\theta = \cos(-\theta)$ to simplify the integrand (or use $\sin\theta = -\sin(-\theta)$ to simplify the result) to obtain
$3\sin(2t) - 2\cos(4t) + C$.

(b) For $-3 < t < 3$,

$$\int \frac{1}{\sqrt{9-t^2}}\,dt = \arcsin\left(\tfrac{1}{3}t\right) + C.$$

(c) For $t < 0$,

$$\int \frac{5t^3+7}{t}\,dt = \int \left(5t^2 + \frac{7}{t}\right)dt = \tfrac{5}{3}t^3 + 7\ln(-t) + C.$$

(d) For $t > 0$,

$$\int \left(2\ln(4t) - \frac{2}{t}\right)dt = 2t(\ln(4t) - 1) - 2\ln t + C.$$

(e) Choose $a = 1$, $b = -1$. Then $b < x < a$ and we can use the standard integral

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\ln\left|\frac{a-x}{x-b}\right| = \frac{1}{a-b}\ln\left(\frac{a-x}{x-b}\right).$$

So

$$\int \frac{1}{(x-1)(x+1)}\,dx = \frac{1}{2}\ln\left(\frac{1-x}{x+1}\right) + C.$$

Solution to Exercise 39

Using $v^2 - k^2 = (v-k)(v+k)$, and taking $a = k$ and $b = -k$ in the Handbook entry for $\displaystyle\int \frac{1}{(x-a)(x-b)}\,dx$, with $b < x < a$, we obtain

$$\begin{aligned}\int \frac{1}{v^2-k^2}\,dv &= \int \frac{1}{(v-k)(v+k)}\,dv \\ &= \frac{1}{2k}\ln\left|\frac{k-v}{v+k}\right| + C \\ &= \frac{1}{2k}\ln\left(\frac{k-v}{v+k}\right) + C \quad (-k < v < k).\end{aligned}$$

Solution to Exercise 40

(a) If $u = 2 + 4y^3$, then $du/dy = 12y^2$. So

$$\begin{aligned}\int y^2 \exp(2+4y^3)\,dy &= \tfrac{1}{12}\int e^u \frac{du}{dx}\,du \\ &= \tfrac{1}{12}\int e^u\,du \\ &= \tfrac{1}{12}e^u + C = \tfrac{1}{12}e^{2+4y^3} + C.\end{aligned}$$

(b) If $u = \sin y$, then $du/dy = \cos y$. So

$$\begin{aligned}\int \cos y \sin^2 y\,dy &= \int u^2 \frac{du}{dy}\,dy \\ &= \int u^2\,du \\ &= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3 y + C.\end{aligned}$$

(c) If $u = 1 - t^2$, then $du/dt = -2t$. So

$$\begin{aligned}\int t\sqrt{1-t^2}\,dt &= -\tfrac{1}{2}\int \sqrt{u}\,\frac{du}{dx}\,dx \\ &= -\tfrac{1}{2}\int u^{1/2}\,du \\ &= -\tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right) + C \\ &= -\tfrac{1}{3}\left(\sqrt{1-t^2}\right)^3 + C.\end{aligned}$$

(d) If $u = 1 + x^2$, then $du/dx = 2x$. So

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx \\ &= \tfrac{1}{2}\int \frac{1}{u}\,du \\ &= \tfrac{1}{2}\ln(u) + C = \tfrac{1}{2}\ln(1+x^2) + C.\end{aligned}$$

In this case, there is no problem with the domain of the ln function because $u = 1 + x^2 > 0$. The answer can also be obtained directly from equation (82) because the integrand is proportional to $g'(x)/g(x)$ with $g(x) > 0$.

(e) If $u = 1 + \sin^2 t$, then $du/dt = 2\sin t\cos t = \sin 2t$, using the trigonometric identity for $\sin 2t$ (equation (41)). So the integrand is of the form $g'(x)/g(x)$ with $g(x) > 0$. Hence equation (82) gives

$$\int \frac{\sin 2t}{1+\sin^2 t}\,dt = \ln|1+\sin^2 t| + C = \ln(1+\sin^2 t) + C.$$

(f) Using equation (82) with $g(y) = 1 - y^2$, so that $g'(y) = -2y$, we have (for $y \neq \pm 1$)

$$\begin{aligned}\int \frac{y}{1-y^2}\,dy &= -\tfrac{1}{2}\int \frac{-2y}{1-y^2}\,dy \\ &= -\tfrac{1}{2}\ln|1-y^2| + C.\end{aligned}$$

Solution to Exercise 41

Taking $u = 3x + 1$, we have $du/dx = 3$, so we can take $du = (du/dx)\,dx = 3\,dx$, and $dx = du/3$. Also, $x = (u-1)/3$, so

$$9x^2 + 1 = (u-1)^2 + 1 = u^2 - 2u + 2.$$

Putting everything together gives

$$\begin{aligned}I = \int \frac{9x^2+1}{3x+1}\,dx &= \int \frac{u^2-2u+2}{u}\,\frac{du}{3} \\ &= \tfrac{1}{3}\int \left(u - 2 + \frac{2}{u}\right)du \\ &= \tfrac{1}{3}\left(\tfrac{1}{2}u^2 - 2u + 2\ln u\right) + C \\ &= \tfrac{1}{6}(3x+1)^2 - \tfrac{2}{3}(3x+1) + \tfrac{2}{3}\ln(3x+1) + C.\end{aligned}$$

There is no problem with the domain of the ln function in this case because $3x + 1 > 0$.

Solution to Exercise 42

(a) Take $f(x) = x$ and $g'(x) = e^{-x}$. Then $f'(x) = 1$ and $g(x) = -e^{-x}$. So

$$\begin{aligned}\int xe^{-x}\,dx &= x(-e^{-x}) - \int 1 \times (-e^{-x})\,dx \\ &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + C \\ &= -(x+1)e^{-x} + C.\end{aligned}$$

(b) Take $f(x) = x^2$ and $g'(x) = e^{-x}$. Then $f'(x) = 2x$ and $g(x) = -e^{-x}$. So

$$\begin{aligned}\int x^2e^{-x}\,dx &= x^2(-e^{-x}) - \int 2x(-e^{-x})\,dx \\ &= -x^2e^{-x} + 2\int xe^{-x}\,dx.\end{aligned}$$

Using the result of part (a), we get

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2(-(x+1)e^{-x} + C) \\ &= -(x^2+2x+2)e^{-x} + B,\end{aligned}$$

where $B = 2C$ is an arbitrary constant.

(c) Take $f(x) = \sin x$ and $g'(x) = e^x$ (the other way around also works). Then $f'(x) = \cos x$ and $g(x) = e^x$. So

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

We now evaluate the integral on the right-hand side by parts, taking $f(x) = \cos x$ and $g'(x) = e^x$. Then $f'(x) = -\sin x$ and $g(x) = e^x$. So

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx.$$

Putting the two results together, we get

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx,$$

and rearranging gives

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x).$$

Solution to Exercise 43

(a) $\displaystyle\int_0^1 (x^3-2)\,dx = \left[\tfrac{1}{4}x^4 - 2x\right]_0^1 = \left(\tfrac{1}{4} - 2\right) - (0-0) = -\tfrac{7}{4}.$

(b) $\displaystyle\int_1^2 (x^3-2)\,dx = \left[\tfrac{1}{4}x^4 - 2x\right]_1^2 = \left(\tfrac{16}{4} - 4\right) - \left(\tfrac{1}{4} - 2\right) = \tfrac{7}{4}.$

(c) $\displaystyle\int_0^2 (x^3-2)\,dx = \left[\tfrac{1}{4}x^4 - 2x\right]_0^2 = \left(\tfrac{16}{4} - 4\right) - (0-0) = 0.$

This integral is the sum of the integrals in parts (a) and (b).

Solution to Exercise 44

$$\begin{aligned}\int_0^{3/2} \frac{1}{9+4z^2}\,dz &= \tfrac{1}{4}\int_0^{3/2} \frac{1}{9/4+z^2}\,dz\\ &= \tfrac{1}{4}\left[\frac{1}{3/2}\arctan\left(\frac{x}{3/2}\right)\right]_0^{3/2}\\ &= \tfrac{1}{6}(\arctan 1 - \arctan 0)\\ &= \tfrac{1}{6}\left(\tfrac{\pi}{4}-0\right) = \tfrac{\pi}{24}.\end{aligned}$$

Solution to Exercise 45

Make the change of variable $u = 3\pi x$, so that $x = u/3\pi$ and $du = (du/dx)\,dx = 3\pi\,dx$. Expressing the limits of integration in terms of u, we see that the lower limit is $u = 0$, while the upper limit is $u = 3\pi$. Hence

$$\begin{aligned}I &= \int_{u=0}^{u=3\pi} \left(\frac{u}{3\pi}\right)^2 \sin^2 u \left(\frac{du}{3\pi}\right)\\ &= \left(\frac{1}{3\pi}\right)^3 \int_{u=0}^{u=3\pi} u^2 \sin^2 u\,du.\end{aligned}$$

We can then use the given integral to conclude that

$$I = \left(\frac{1}{3\pi}\right)^3 J_3 = \left(\frac{1}{3\pi}\right)^3 \left(\frac{3^3\pi^3}{6} - \frac{3\pi}{4}\right) = \frac{1}{6} - \frac{1}{36\pi^2}.$$

Solution to Exercise 46

(a) First notice that $f(x) = e^{-x^2}\sin(x)$ is an odd function of x because

$$f(-x) = e^{-(-x)^2}\sin(-x) = -e^{-x^2}\sin(x) = -f(x).$$

Since the range of integration (from $-\pi$ to $+\pi$) is symmetric about the origin, the integral vanishes.

(b) The integrand is odd because

$$f(-x) = -x\sin(1+(-x)^4) = -f(x).$$

Since the range of integration is symmetric about the origin, the integral vanishes.

(c) We write the integral as

$$I = \int_{-1}^{1} x^2\cos(x^3)\,dx + \int_{-1}^{1} x^3\cos(x^3)\,dx.$$

The second integral vanishes since the integrand is odd and the range of integration is symmetric about the origin.

The first integral can be evaluated by changing the variable to $u = x^3$, so that $du = 3x^2\,dx$, and the two limits become $u = -1$ and $u = 1$.

Hence

$$I = \tfrac{1}{3}\int_{u=-1}^{u=1} \cos u\,du = \left[\tfrac{1}{3}\sin u\right]_{u=-1}^{u=1} = \tfrac{2}{3}\sin(1) \simeq 0.561.$$

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figure 1: NASA.

Figure 2: Taken from www.astro.le.ac.uk/users/gwy/images/nano.

Figure 23: Taken from http://wrongsideofmemphis.com/tag/kepler.

Figure 24: M.V. Berry and S. Klein (1996) *Coloured Diffraction Catastrophes*, National Academy of Sciences.

Figure 32(b): Image Editor / www.flickr.com.

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